A COURSE IN ARITHMETIC FOR TEACHERS AND TEACHER-TRAINING CLASSES

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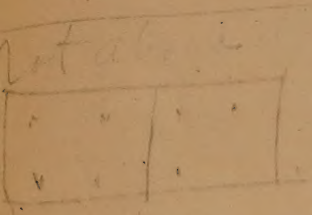

$3 \times 5 = 15$

$7 + 4 = 11$

$11 + 7 = 18$

A COURSE IN ARITHMETIC FOR TEACHERS AND TEACHER-TRAINING CLASSES

DISCARD

By

JAMES ROBERT OVERMAN, A. M.

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PREFACE

The author's previous book, Principles and Methods of Teaching Arithmetic, dealt with fundamental principles and methods of procedure. The present volume provides a thorough review of subject matter combined with suggestions for teaching and a brief treatment of the history of arithmetic. The two books together cover the ground outlined in the author's Syllabus of Courses in Arithmetic as Preparation for Teaching in the Elementary Schools that was awarded first prize in the competition held by the National Council of State Normal School Presidents and Principals. It is believed that the two books provide a complete and thorough preparation for the teaching of arithmetic and that they will be found suitable for use as texts in teacher-training institutions and by teachers desiring to review and extend their knowledge of the subject and to study modern methods of teaching.

The first requisite for successful teaching of arithmetic is a thorough mastery of the subject extending somewhat beyond the portions to be taught. An advanced course in arithmetic is therefore a necessary part of the preparation of every teacher of the subject in the elementary schools. The usual formal texts in higher or advanced arithmetic are not written primarily for use in such courses and do not meet the needs of teachers. The present book is the result of the author's experience in the preparation of teachers of arithmetic and is planned especially to meet the needs of such teachers. All of the material of the book has been developed and tested in the class room.

The following are the distinctive features of this book:

1. It provides a thorough review of the elements of arithmetic and makes clear the underlying principles.

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2. It extends the teacher's knowledge of the subject by the inclusion of such topics as short methods, methods of checking, accuracy of results, graphical representation of statistics, educational measurements, budgets, accounts, investments, intuitive geometry, simple algebra, etc.

3. It combines with a study of subject matter a brief discussion of methods of teaching. It is the author's firm belief that the best place to study the details of method is in connection with a review of subject matter.

4. A brief treatment of the history of arithmetic is a valuable portion of the book. A knowledge of the history of the subject makes it more interesting to the teacher and provides material that she can use in her own classes to increase the interest of the pupils. It also gives the teacher a broader background and makes her more open minded.

5. The book is written from the modern social point of view and it is believed that it will do much to instill this point of view into the teacher.

The author wishes to acknowledge his indebtedness to several of his colleagues who have tested the material of the book in their classes and to his wife for her assistance in correcting the proof.

JAMES ROBERT OVERMAN

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ARITHMETIC FOR TEACHERS

CHAPTER I.

READING AND WRITING NUMBERS

I. Introduction.

1. The Origin and Development of the Number Idea. Some knowledge of the history of arithmetic is necessary for the full understanding of the subject as it is today. The first step in the development of arithmetic was the origin of the idea of number. Primitive number ideas are probably as old as the human race, at least the earliest peoples of whom we have any record all had some idea and knowledge of numbers, and the same is true of the most primitive savages of modern times. There are several psychological theories that attempt to account for the origin of the number idea but history throws no light on this question.

Pupils when they enter school usually have certain number ideas. It is the duty of the teacher in the primary grades to take this knowledge of the pupils and extend and clarify it so as to form a firm and dependable foundation for the future work in numbers. The number idea is complex and can only be acquired as the result of wide and varied experience. The complete concept of any number, such as six, involves the notion of position in a series, as the sixth pupil in a row; the idea of the number in a group, as six pencils; the idea of magnitude, as a line six inches long or an object that weighs six pounds; and a knowledge of the relations of the given number to other numbers, as six is one more than five and one

less than seven, two more than four and two less than eight, is made up of two threes or three twos, etc.

The pupils should first get the idea of number as meaning position in a series. This is obtained through counting. Much practice should be given in counting objects of various sorts without any attempt at first to have the pupils recognize any of these numbers of things. Fortunately, children like to count for the sake of counting and this natural liking should be taken advantage of in teaching numbers. Counting of this kind should go considerably beyond the pupils' present number needs.

Next the pupils must get the idea of number as answering the questions "How many?" and "How much?". By counting groups of objects the pupils should get such a definite idea of two, three, and four that they can recognize groups of two, three, and four objects at sight without stopping to count, and can recognize larger groups by breaking them up into groups of twos, threes, and fours. The "how much" idea is developed through measuring and the pupils should be given plenty of practice in this kind of work, using all of the common standard units of measure.

A number is not completely known until it is known in all of its relations to the other numbers. This knowledge should be gained by making comparisons of objects and groups of objects. Thus, by comparison, the pupils find that there are three more objects in a group of nine than in a group of six, and that a line twelve inches long is just twice as long as one six inches long, etc.

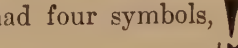
2. Primitive Methods of Counting. Among primitive peoples, *counters* of some kind were usually used in counting. These counters were sometimes sticks, pebbles, shells, notches in a stick, or knots tied in a string but more often they were simply the ten fingers. In counting a group containing more than ten objects, the usual practice was to count to ten on

the fingers and then to make a record of the ten in some way, such as placing a pebble on the ground. The count was then continued on the fingers, and when ten was reached again a second pebble was placed with the first. Thus each pebble represented ten and each finger one, so two pebbles and three fingers meant two tens and three units, or twenty-three.

3. Methods of Writing Numbers. Practically all civilized nations today use the same method of writing numbers. Before the present method became generally known, each civilized nation and people had its own system. Two of these older systems are considered here, as a knowledge of them will enable the teacher to better understand and appreciate the one now in use.

II. The Babylonian System

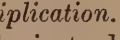
4. History. One of the earliest civilizations of which we have any knowledge is that of the Babylonians, who lived on the east bank of the Euphrates River. The Babylonians wrote on moist clay with a pointed stick. In writing they pressed the sharp end of the stick into the clay making a wedge or arrow-shaped mark. For this reason their inscriptions are often called *cuneiform* writing, a word meaning "wedge-shaped." When a permanent record was desired the clay tablets were baked in the hot sun. Many of these are still in existence today. The Babylonian method of writing numbers was in use at least as early as 2300 B. C. and perhaps much earlier.

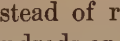
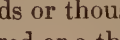
5. Symbols. The system had four symbols, stood for unity, for ten, for one hundred, and  for a thousand.

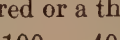
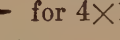
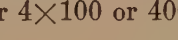
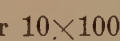
6. Principles. Three principles were used in writing numbers.

I. Repetition. In writing a number of units or tens the unit or ten symbol was *repeated* the required number of

times, as for 3, for 6, and for 40.

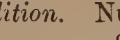
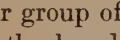
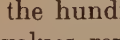
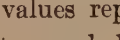
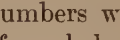
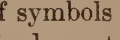
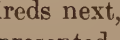
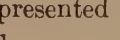
II. Multiplication. In writing any number of hundreds or thousands, instead of repeating, the symbols representing the number of hundreds or thousands were placed to the left of the symbol for a hundred or a thousand and used as a *multiplier,* as for 4×100 or 400, for 3×1000 or 3000, and  for 10×1000 or 10,000.

III. Addition. Numbers were written from left to right the symbol or group of symbols representing thousands being written first, the hundreds next, the tens next, and the units last, and the values represented by the thousands, hundreds, tens and units symbols were *added;* as for 11,  for 111 and  for 324.

7. Numbers Written in the Babylonian System. A careful study of the principles given above and of the following numbers should make the Babylonian system of writing numbers clear to the reader.  equals 500+20+7=527,  =40+5=45,  =1000+300+10+4=1,314, and  =4000+300+20+5=4,325.

Exercise. Babylonian Notation

1. Read the following:

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 

2. Write the following in the Babylonian notation:

1.	37	5.	781
2.	82	6.	936
3.	549	7.	3,847
4.	105	8.	7,592

III. The Roman System

8. Present Use and History. Of the older systems the only one that is in use today is the Roman system. It is still used occassionally on clocks, in outlines, in numbering chapters in books, and in the inscriptions of dates on buildings.

The origin of the Roman numerals is uncertain but they were probably inherited from the Etruscans. The Roman notation was used by the Romans and throughout Europe during the middle ages.

9. Symbols. The symbols in the Roman notation *as it is used to-day* are as follows:

I	represents *one*	L	represents *fifty*
V	represents *five*	C	represents *one hundred*
X	represents *ten*	D	represents *five hundred*

M represents *one thousand*

The ancient Roman and medieval European symbols differed in some cases from those in modern use. In early times five was written either or , fifty was written or , one hundred was written , five hundred was written or , and one thousand was written . In later times, in order to lessen the number of necessary symbols, the number symbols were modified so as to look like letters. The student need not attempt to remember any of the older forms.

10. Principles. Numbers are written from left to right, thousands first, then hundreds, tens and units. The following principles were used by the Romans and are used today.

I. Repetition. Repeating a symbol repeats its value, as II for 2, XXX for 30, and CC for 200.

II. Addition. When a symbol or group of symbols of lesser value is placed *after* one of greater value their values are *added,* as XII for 12, VI for 6 and LV for 55.

In late Roman and in medieval times two other principles were occassionally used. Neither of these came into general use, however, until modern times.

III. Subtraction. When a symbol of lesser value is placed *before* one of greater value, its value is *subtracted* from that of the symbol of greater value, as IV for 4, IX for 9, XL for 40, and CM for 900.

IV. Multiplication. A bar drawn over a symbol or group of symbols, multiplies the value by 1000, as $\overline{X}$ for 10,000, and $\overline{CCXXVII}$ for 227,000.

11. Numbers Written in the Roman Notation.

CCCLXXVI=300+70+6=376

MDCCCCXVIIII or MCMXIX=1000+900+10+9=1919

MCCCCLXXXXII or MCDXCII=1000+400+90+2=1492

EXERCISE. ROMAN NOTATION.

1. Read the following:

1. CCLVII
2. MMDCCLXXXXIIII
3. IIDCCXCIV
4. MDCCLXXVI
5. DCCCCLXXXXVIIII
6. CMXCIX
7. $\overline{XXXIV}$CCLXXXIV
8. MDCCCLXXXVIII

2. Write the following in the Roman notation using repitition, and addition only.

1. 274
2. 949
3. 1921
4. 9944

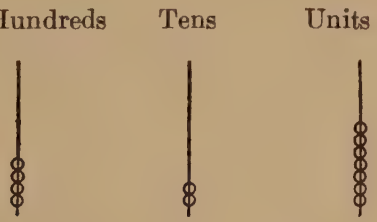
3. Write the numbers given in Ex. 2 in the Roman notation using repitition, addition and subtraction.

4. Write the following numbers in the Roman notation, using all four principles.

1. 7,894
2. 18,756
3. 2,897,479

IV. Defects of Babylonian and Roman Systems

12. The Abacus. The chief defect of the Babylonian and Roman and of all the older systems was that they were of little assistance in computation. In adding, subtracting, multiplying and dividing the written symbols were used in recording the original numbers and the result, but the actual calculation was not performed with the written symbols but on an *abacus.* In its primitive form the abacus consisted of a table covered with sand in which a series of parallel lines were drawn. This sand table was the form used by the Hindus and the early Greeks. By the Greeks it was called an *abax* meaning "dust-board" and from this name comes our word *abacus.* Movable counters were placed on the lines to represent units, tens, hundreds, etc. Thus 427 would be represented on this primitive form of abacus as follows:



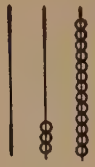
One form of Roman abacus was made of bronze with parallel slots in which were movable counters. The modern abacus of China, Russia and Japan consists of movable beads or counters strung on parallel wires stretched across a wooden frame. The *number frame* often used in primary arithmetic is a form of abacus.

13. Addition on the Abacus. In the middle ages in Europe the addition of XXXVII and LXXXVI was performed on the abacus in a manner similar to the following:

First Step

Place seven counters on the first line to represent VII in XXXVII and three on the second to represent XXX.

Second Step



Place six more counters on the first line to represent the VI in LXXXVI.

Third Step

Take ten counters from the first line and "carry" one of them to the second.

Fourth Step

Place eight more counters on the second line to represent the LXXX in LXXXVI.

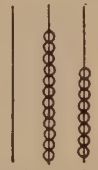
Fifth Step

Take ten counters from the second line and "carry" one to the third. Result CXXIII.

14. Subtraction on the Abacus. The subtraction of XXV from LII is shown.

First Step	Second Step	Third Step
Place two counters on the first line to represent the II of the LII, and five on the second to represent the L.	"Borrow" one counter from the second line (one ten) and substitute for it ten on the first (ten units).	Take five counters from the first line for the V of XXV and two from the second for the XX. Result XXVII.

15. Multiplication on the Abacus. Multiplication on the abacus was long and difficult and as a result many methods were devised in the attempt to find a short one. One method, that of repeated doubling, is shown in the following in which XXXII is multiplied by IX:

First Step	Second Step	Third Step	Fourth Step
			
Place two counters on the first line to represent the II of the XXXII and three on the second to represent the XXX.	Double the original number by putting two more counters on the first and three more on the second. Record this result as LXIIII.	Double the LXIIII in the same way.	Take ten counters from the second line and substitute one on the third. Record this result as CXXVIII.

Fifth Step	Sixth Step	Seventh Step
Double again.	Take ten counters from first line and substitute one on second. Record this result as CCLVI. This is eight times the original number as it has been doubled three times.	Add the original number XXXII by adding two counters to the first line and three to the second. This result CCLXXXVIII is the required product, or IX times XXXII.

16. Division on the Abacus. Division on the abacus was even more difficult than multiplication and except in simple cases could be performed only by a few scholars. The following method of repeated subtraction was sometimes used for small numbers. The example illustrated is CCCXVII divided by LXXXIIII:

First Step	Second Step	Third Step	Fourth Step
Represent the dividend CCCXVII.	Subtract LXXXIIII, leaving CCXXXIII.	Subtract LXXXIIII, again leaving CXXXXVIIII.	Subtract LXXXIIII again, leaving LXV.

As LXXXIIII was subtracted III times and there is LXV left, CCCXVII divided by LXXXIIII gives a quotient of III with a remainder of LXV.

The foregoing illustrations are sufficient to show how greatly the world was handicapped in arithmetic before the general introduction of the present system, by the lack of a usable method of writing numbers. With practice, addition and subtraction could be performed rapidly and accurately on the abacus, but multiplication and division were both long and difficult.

Exercise. Calculation on the Abacus.

Make an abacus by drawing parallel lines on a large piece of paper. Use any convenient objects for counters. Perform the following calculations.

1. Add:

	1.	374	2.	568
		879		374
				563

2. Subtract:

	1.	756	2.	934
		284		275

3. Multiply:

	1.	13×42	2.	25×323

4. Divide:

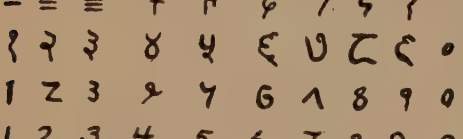
	1.	$329 \div 72$	2.	$361 \div 45$

V. The Hindu-Arabic System

17. History. The system in general use at the present time originated in central India about 500 B. C. At that time the system was incomplete as it did not contain a symbol for zero and this was not added until a thousand years later, or about 500 A. D. Shortly after the invention of the zero symbol the system became known to Arab scholars and was carried by

them into Northern Africa and Spain. European scholars and merchants were acquainted with this system as early as the tenth century A. D., but it did not come into general use in Europe until the sixteenth century. By the seventeenth century the use of the Hindu system had become practically universal throughout Europe.

18. Symbols. The modern symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, vary considerably from the Hindu, Arabic and medieval European forms as is shown in the following table.

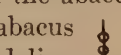


19. Place Value. The fundamental difference between this system and the Babylonian and Roman is that it is a *place value* system and the others were not. Each symbol not only has a certain value in and of itself as in the other systems but also has another value due to the *place* in which it is written.

In assigning values to the various places the Hindus started at the right. Any symbol written in the first place means that number of units, in the second place that number of tens, in the third hundreds, etc., the value of each place to the left being ten times that of the preceding place. In the Roman system V always stood for five *units* no matter where it was written in the number, as in XV and LVIIII. In the Hindu-Arabic system, however, 5 may mean five units, five tens, or five hundreds, etc. according to the place in which it is written as in 35, 57, 582, etc.

Since the value of each place, going from right to left, is ten times that of the preceding place, this system is said to have ten as a *base* and is therefore called a *decimal system*. It would be possible to have a place value system with any number as a base. The choice of ten was probably accidental and arose

from the method of counting on the fingers previously described.

20. The Zero. As has been shown the Babylonians and Romans both used place value in representing a number on the abacus. A counter placed on the first line meant one unit, on the second line one ten, etc. They could not use place value in their written systems, however, because they did not have a symbol to represent nothing and did not see the necessity for such a symbol. The Hindus were the first to realize this need and their greatest contribution to arithmetic was the invention of a symbol to stand for nothing or absence of value. Without such a symbol place value would be possible on the abacus but not in the written symbols. Thus, on the abacus  stands for three hundred two. The unoccupied line shows that there are no tens and that the three is, therefore, three hundreds. In the written symbols, 302, the symbol for nothing, or zero, serves the same purpose as the unoccupied line on the abacus. Without the zero it would be impossible to write numbers such as three hundred two or twenty in this way as they would appear to be thirty-two and two respectively.

21. Reading Numbers in the Hindu-Arabic System. In France and America today the places, starting at the right, are named as follows.

Trillions	Hundred-billions	Ten-billions	Billions	Hundred-millions	Ten-millions	Millions	Hundred-thousands	Ten-thousands	Thousands	Hundreds	Tens	Units
x ,	x	x	x ,	x	x	x ,	x	x	x ,	x	x	x

A thousand thousand is called a *million,* a thousand million is called a *billion,* a thousand billion is called a *trillion,* etc. As an aid in reading, large numbers are often separated into

periods of three digits each beginning at the right, by means of commas or spaces. The first period at the right is units, the next thousands, the next millions, etc. The symbols in each period are read as if the period stood alone and then the name of the period is added, except in the case of units period. The word "and" should not be used in reading whole numbers. Thus 9,384,576 should be read "nine million, three hundred eighty-four thousand, five hundred seventy-six."

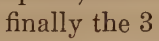
In England the places are named somewhat differently. According to the English system it takes a million million to make a billion, and a million billion to make a trillion. The names of the places in the English system are given below.

Billions		Hundred-thousand-millions	Ten-thousand-millions	Thousand-millions	Hundred-millions	Ten-millions	Millions		Hundred-thousands	Ten-thousands	Thousands	Hundreds	Tens	Units
X	,	X	X	X	X	X	X	,	X	X	X	X	X	X

In business and other practical affairs it is quite common to read a number by reading the digits in order, singly or in pairs, with no reference to place value. Thus 3897 is read "three, eight, nine, seven" or "thirty-eight, ninety-seven."

22. Advantages of the Hindu-Arabic System. The Hindu-Arabic system is better than the other systems in several ways. Its advantages are all the direct result of its place value feature. It has already been seen that the other systems could not be used in calculation. In the Hindu-Arabic system all numbers are written as so many units, tens, hundreds, etc. This makes it possible to break large numbers into parts and deal with the parts separately. As there can never be more than nine units, tens, or hundreds in any number, by memorizing the addition, subtraction, multiplication and division

facts for all numbers from one to nine, it is possible to add, subtract, multiply and divide numbers of any size with the aid of the Hindu-Arabic symbols and without the aid of the abacus. Thus it is possible to multiply 323 by 3 by breaking the 323 into its parts, first multiplying the 3 units by 3, then the 2 tens, and finally the 3 hundreds. The greatest advantage of the Hindu-Arabic system is, therefore, the fact that it makes possible arithmetical computation as we know it today.

This system also has three other advantages. (*a*) It usually takes fewer symbols to write a given number. Thus thirty-seven requires two symbols in this system (37), ten in the Babylonian (), and six in the Roman (XXXVII). (*b*) The system is unlimited. The Babylonians and Romans both had difficulty in writing large numbers but numbers of any size can be written in the Hindu-Arabic system without the addition of new symbols or principles. (*c*) It makes possible an easier way of writing fractions (decimals), and as a result greatly simplifies computations with fractions. The world has not yet taken full advantage of this.

Exercise. Place Value

1. Explain the use of the zeros in the following numbers: 50, 706, 900, 3890, 750.

2. If zero is annexed to the right of 786 how is the value of each digit affected? How is the value of the whole number affected? Answer the same questions if two zeros are annexed. Explain each answer.

3. If zero is placed to the left of 786 how is the value of each digit affected? How is the value of the whole number affected? Explain.

4. State a rule for multiplying an integer by 10. By 100. Explain each.

5. If a 4 is placed to the left of 786 how is the value of the number changed? Explain.

6. If a 4 is placed to the right of 786 how is the value of the number changed? Explain.

7. What would be the easiest way to multiply a number by 10 and add 3? Multiply by 100 and add 77? Explain.

8. If one zero is taken away from the right of 78900 how is the value of each digit affected? Explain. How is the value of the whole number affected? Explain. Answer the same questions for two zeros.

9. If the 9 is erased in 789 how is the value affected? Explain.

10. If 3578 is divided by 10 what would the quotient and the remainder be? Explain. Answer the same questions for 100.

11. State a rule for obtaining the quotient and remainder when an integer is divided by 10. By 100. Explain each.

12. A place value system having twelve as base is called a duodecimal system. What would be the values of the first three places in this system?

13. In the duodecimal system it is necessary to have two more symbols than in the decimal system, namely a symbol for ten and one for eleven. Explain.

14. The small 12 printed to the right of and below the following numbers indicates that they are written in the duodecimal system. T is used to represent ten and E to represent eleven. Change to the decimal system.

a. $234_{12}=2\times144+3\times12+4$
$=288+36+4=328$

b. 10_{12} *e.* T_{12}

c. 325_{12} *f.* $2E_{12}$

d. 41_{12} *g.* 300_{12}

15. Write the following in the duodecimal system.

a. $32=2\times12+8=28_{12}$

b. 11 *d.* 40

c. 160 *e.* 131

16. How is the value of a number written in the duodecimal system, affected by annexing a zero to the right? Explain.

17. State a rule for multiplying a number, written in the duodecimal system, by 144. Explain.

CHAPTER II.

ADDITION

I. The Meaning and Use of Addition

23. The Addition Idea. It is neither necessary nor desirable that pupils in the grades be able to give a formal definition of the process of addition. It is very important, however, that they, through their first experience with addition, get a clear idea of the *meaning* and *use* of the process. This can only be obtained through concrete experience with simple addition situations. Thus the pupils meet a situation such as going to the play toy-store and buying a ball for 3 cents and a horn for 5 cents. By using toy money and counting they find that they must pay 8 cents altogether. They are told that this process of combining 3 cents and 5 cents is called addition. Through experiences similar to this, in which the term addition is constantly employed, the pupils come to look upon addition as a "putting-together" or "combining" process, and to recognize any situation or problem that involves the combining of numbers as calling for the process of addition.

24. Definitions. Although formal definitions have little or no place in the grades they do serve a useful purpose in a book for teachers. They enable the teacher to test out the definiteness and correctness of her own ideas. It is suggested that the teacher before reading the definitions given in this book formulate in definite statements her own ideas and then compare them to the definitions given. The definitions given are also useful for reference and as a basis for some of the theory that follows.

The process of finding, without counting, a single number or quantity, equal in value to the combined values of two or more other numbers or quantities is called *addition*.

The word *add* is from the Latin *addere,* meaning to give to.

Latin arithmetics spoke of the numbers to be added as *numeri addendi.* This was abbreviated to addendi and translated into English as *addends.*

In America the word *sum* is usually used for the answer in addition. In practical affairs the result is often called the *total.*

II. Principles of Addition.

25. Principles. There are certain fundamental laws or principles underlying the various arithmetical processes, with which the pupils must become familiar at an early date. There are two that are particularly important in addition.

I. Only quantities having the same name (expressed in the same units) can be added or combined, and the result always has the same name as the addends.

2¢	2 lbs.	2 lbs.
3¢	3 lbs.	3¢
5¢	5 lbs.	Impossible

A knowledge of this principle is necessary to an understanding of many things in arithmetic. Thus in adding numbers units must be placed under units, tens under tens, etc. as

27
8
394

Also it follows from this principle that fractions must be changed to the same denominator before adding and that $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$ and not $\frac{5}{14}$, an error that pupils will often make.

II. The sum is independent of the order in which the addends are combined. $2+3=3+2$; $7+8+9=7+(8+9)=(7+9)+8$, etc; $3+2+5+8=(3+2)+(5+8)=(3+5+8)+2$, etc.

A knowledge of this principle will save much time in learning the addition facts or combinations. The pupils discover that $3+2=5$ and that $2+3=5$. From this and similar cases

they should be lead to generalize as soon as possible so that when they learn a new fact as 7+8=15 they can with no extra drill also learn 8+7=15 by regarding the two as a single fact and drilling on both forms.

This principle is also made use of in checking addition and in certain short methods of adding as will be shown later.

III. The Process of Addition

26. Old Methods of Adding. Little change has occurred in the method of performing addition since the invention of place value. The method used by the Hindus as described in the works of Bhaskara (12th century) was substantially the same as our present method. This book of Bhaskara is our chief source of information concerning the methods of calculation of the Hindus. They wrote upon a board, less than a foot square, strewn with sand, on which they wrote the figures with a small stick. To be legible the figures had to be quite large and as the board was small they erased each figure, by smoothing it over with the thumb, as soon as it had served its purpose.

In adding 575 and 968 they wrote the 968 under the 575 and thought 8+5=13 erasing the 8 and 5 and writing the 3 in units place and the 1 in tens, leaving the following arrangement of figures on the board.

57
96
13

They then thought 1+6+7=14, erasing the 1, 6 and 7, and writing the 4 in tens place and the 1 in hundreds leaving on the board the following arrangement.

5
9
143

Then 1+9+5=15, erasing the 1, 9 and 5, and writing the 5 in hundreds place and the 1 in thousands. When they were

through nothing remained on the board but the result 1543.

The Hindus also had a similar method of adding in which they started at the left instead of at the right.

With the invention of paper, erasing became more difficult so in Europe (15th and 16th centuries) they crossed out the figures instead of erasing them. This was called by the English the "Scratch Method". The work usually started at the right as we do today but the result was usually put above. The addition of 575 and 968 by the scratch method appears as follows.

54
1113
~~575~~
~~968~~

Gradually the crossing out of the figures was abandoned as it was not necessary and the figures carried, instead of being written, were kept in the head, giving us the process of addition as we know it today.

EXERCISE. OLD METHODS OF ADDING.

Add the following using the "Scratch" method:

1.	2.	3.
347	7569	837
845	4627	568
		346

27. The Present Method of Adding. Pupils should be taught *from the beginning* to think results only. Thus in adding the column given, the pupils should not think 8 and 7 are 15, 15 and 6 are 21, 21 and 8 are 29, 29 and 7 are 36, but should think only 15, 21, 29, 36. The number carried should be added immediately to the first figure in the next column, thinking 12, 19, 23, 28, 36. Some people prefer to start at the top instead of at the bottom.

387
758
846
477
898

In the case of long columns it is sometimes best to write down the number carried instead of immediately adding it to the next column. This enables you to resume the work where you left off if interrupted and also is convenient in checking as

a mistake in one column can be corrected without adding all of the columns again.

Several methods of doing this are in common use. One of the best is to record the sum of each column separately and then add these partial sums to obtain the final result.

387
758
846
477
898
36
33
30
3366

28. An Analysis of the Addition Process with Suggestions for Teaching. "The four fundamental processes of addition, subtraction, multiplication, and division are not simple processes that the pupils can learn all at one time, but are very complex and must be presented gradually, one step at a time. A mastery of any one of them involves the ability to do many different things, demands not a single ability but many different abilities and habits. It is very important that all of these abilities and habits be systematically developed one by one and that none be neglected. In order to do this the various difficulties and habits involved in the complete process must be carefully separated and each must be mastered and mechanized before the next is introduced."*

The following analysis shows some of the difficulties to be mastered and habits to be formed in the process of addition. These are arranged approximately in the order in which they are met by the pupils, but the exact order varies in different texts and schools. Each point should first be understood and then reduced to habit through drill.

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*Overman, Principles and Methods of Teaching Arithmetic.

I. The Addition Facts or Combinations. In the discussion of the Hindu-Arabic notation it was shown that its chief advantage is that numbers written in this system can be readily broken up into parts and the parts handled separately. Thus in adding 976 and 839 we can break the numbers up into units, tens, and hundreds and add the parts separately. In this way numbers of any size may be added if one knows the sum of any two numbers from 0 to 9. These sums are known as the *primary facts of addition* or as the *addition combinations.* The first addition facts must be discovered by the pupils through the counting of objects. Later facts can be obtained from those already known as 7+8 from 7+7 by adding 1. Once discovered they must be memorized through drill. There are in all one hundred of these facts including the thirty-six reverses, such as 3+4=7, the reverse of 4+3=7, and the nineteen having zero as one of the two addends as 0+0=0 and 0+3=3.

These one hundred facts are presented in different schools in various orders. Some schools organize these facts in tables and present the tables in the numerical order as follows:

0	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9
2	2	2	2	2	2	2	2	2	2
2	3	4	5	6	7	8	9	10	11

These are known as the tables of 1's and 2's respectively. There are similar tables of 3's, 4's, 5's, 6's, 7's, 8's and 9's. In many schools the facts in these tables having sums greater than 10 are omitted when the tables are first presented and taken up later after all of the facts having sums less than 10 have been presented. The facts involving 0 are usually not presented with the first few tables but are taken up later.

Some schools make the sum the basis of the organization, teaching in order, the combinations having sums 2, 3, 4, 5, etc., as follows:

1	2	1	3	1	2	4	1	3	2
1	1	2	1	3	2	1	4	2	3
2	3	3	4	4	4	5	5	5	5

In some schools the facts are not organized in tables but are presented as individual facts, the attempt being made to present them in the order of their difficulty. No matter in what order the facts are first presented their order must be constantly changed in the drill work so that each may be learned as a separate fact and not as part of a table.

II. Columns of Three Numbers Each, Sum Less than Ten.

3	2	4
2	2	1
4	5	2

The pupils should think results only from the beginning. In the first example starting at the bottom they should think 6, 9; in the next 7, 9 etc. Work of this kind should start as soon as enough combinations have been learned and much of the drill on the combinations should be obtained by adding columns such as these. Some schools prefer to learn the combinations entirely through work of this kind, claiming that time is saved in this way. Most schools still prefer, however, to give some preliminary drill on the combinations.

III. Addition of Two-place Numbers Without "Carrying."

34	23
12	12
	14

To add these, pupils must learn to break the numbers into their parts and to add the units and then the tens. This is the first place in which the pupils encounter this method of procedure, which is the basis of all of our methods of calculating

with numbers larger than nine. In order to understand this method they must become acquainted with place value to two places. One of the best ways of accomplishing this is to have the pupils represent two-place numbers by means of sticks, using single sticks to represent units and bundles of ten sticks to represent tens. Thus 45 is represented by four bundles of ten sticks each and five single sticks.

The pupils should be able to represent any two-place numbers with sticks and to read and write the number represented by any combination of single sticks and bundles of ten sticks before they take up the addition of two-place numbers.

IV. Addition of Two-place and One-place Numbers Without Carrying.

23	13
2	3
	21

The new habit the pupils must gain here is that of keeping the right hand column straight. They should see in the beginning that this is necessary because of the principle that only numbers having the same names can be added. So it is necessary to keep units under units. Although they should understand the reason the teacher must make certain that they will always do it correctly by forming the proper habit.

V. Zero Difficulties.

36	20	40	20
20	14	16	30
	32	20	10

Even if the zero combinations have been drilled upon, zeros occurring in column addition usually give some trouble and enough drill must be provided on examples involving zeros to overcome this difficulty. Many schools give no drill on zero combinations as such but postpone them until they are met in column addition.

VI. Columns of Three or More Numbers, Sum Greater than Nine, Involving Only a Knowledge of the Combinations.

4	5	6
5	3	2
2	6	1
		2

The new point involved is the question of what to do with the two figures obtained in the answer. The habit of placing the unit figure of the result immediately below the units column should be established from the beginning.

VII. "Addition by Endings" or "Carrying the Combinations to Higher Decades." A knowledge of the hundred primary facts of addition is not sufficient to enable the pupils to add longer columns. In adding the column given at the left the pupils must know, if they start at the bottom, 9+6=15, 15+7=22, 22+5=27, and 27+4=31. The first of these, 9+6=15, is one of the hundred facts, but the others are not although they are closely related to the primary facts. In adding columns pupils must not only know the primary fact 5 but may need to know

4
5
7
6
9

7

15, 25, 35, etc. Much drill must be given on recognizing

7 7 7

the combinations when they are raised to the higher decades. It is not enough to drill on 7, the pupils must also drill on the

4

same fact raised to the higher decades as 17, 27 etc. This is

4 4

also called drill on "addition by endings" as the end figure in 17, 27, etc., is the same as in the primary fact 7. Counting

4 4 4

by 2's, 3's, 4's, etc., beginning with 1, 2, etc., up to the number by which they are counting is excellent preparation for this work. If they count by 4's to 100, starting with 1, then with 2,

with 3, and with 4, the pupils get all possible combinations of 4 with two-place numbers to 100.

VIII. Columns Involving Addition by Endings. In adding the column to the left the pupils should think 13, 22, 26, 34, 37.

3
8
4
9
8
5

IX. Carrying. (a) Carrying one ten, 28 + 17 = 45; 24 + 23 = 47; 39

(b) Carrying two or more tens, 37 + 88 + 19 = 144

In developing the process of carrying, the first example, as 28 + 17 = 45, should be worked out by the pupils by representing the two numbers by means of the sticks. They represent the 28 by two bundles of ten each and eight single sticks and under it 17 by one bundle of ten and seven single sticks. The seven sticks and the eight sticks make fifteen, which they represent by fifteen sticks below the other units. Then one ten and two tens make three tens which they represent by three bundles of ten under the other tens. The pupils then have before them the following:

The teacher next asks what they might do with the 15 single sticks, showing, if necessary, that ten of them can be

made into a bundle of ten and put with the other tens, giving the result

or 4 tens and 5 units, or 45.

Having worked out one or two examples with the sticks the pupils should next try to add these numbers on the blackboard without using the sticks. In doing this they think "7 units and 8 units make 15 units, but 15 units is 1 ten and 5 units, so we have 5 in units place. One ten and 1 ten make 2 tens, and 2 tens make 4 tens." Next the pupils should try to add in the shortest way without thinking about tens and units, as "7 and 8 are 15" (writing the 5 and remembering the 1); "1 and 1 are 2; 2 and 2 are 4" (writing the 4).

Later the carrying of two or more tens can be taken up and can be worked out with the sticks in a similar way if necessary.

X. Addition of Three-place Numbers. (a) Without Carrying.

$$\begin{array}{r} 232 \\ 524 \\ \hline \end{array}$$

As a preparation for this work the pupils must become familiar with place value to three places. Work with the sticks should be given similar to that used for two-place numbers, using a large bundle composed of ten bundles of ten each to represent one hundred.

(b) With carrying from units to tens place.

$$\begin{array}{r} 358 \\ 217 \\ \hline \end{array}$$

(c) With carrying from tens to hundreds place.

$$\begin{array}{r} 273 \\ 142 \\ \hline \end{array}$$

This can be worked out with the sticks in a manner similar to that described above.

(d) With carrying in two places.

457
176

XI. Zero in the sum.

365
237
602

In cases such as this, where the sum of the tens is 10, pupils frequently put down nothing at all in tens place or else put down the 1 of the 10. Plenty of drill must be given on examples with zeros in the sum.

XII. Addition of Addends Having Different Numbers of Digits.

348
27
3849
9

This difficulty is the same as met with in adding two-place and one-place numbers. Sufficient examples of this type must be dictated, or given in the horizontal form 348+27+3849+9 so that the pupils must copy the numbers and form their own column, to thoroughly establish the habit of keeping the units column straight.

XIII. Column Addition with More Than Six Addends.

9
2
4
8
2
7
5
3
6

The new difficulty here arises from the increased attention span. Experiment shows that most pupils can keep their attention fixed on column addition for from six to eight additions. Then the attention wanders and if the pupil is left to his own resources he often gets hopelessly lost, adding so far and no farther, time after time. They must be taught to recognize this difficulty, to keep the proper place in the column with their pencil, and to remember the partial sum while they let their attention wander momentarily. Then when they return to the column they can start where they left off.

EXERCISE. DEVELOPMENT OF CARRYING.

1. Work out in detail, using the sticks, the process of carrying from tens to hundreds place.

IV. CHECKING ADDITION

29. Need of Checks. Methods of checking the fundamental processes should be taught early in the grades and the pupils should be required to check all calculations until the habit of checking is thoroughly established. Even the most accurate calculator will occasionally make mistakes and so some method of detecting errors is necessary. Checking will not only enable pupils to detect errors as soon as made but it will do more than any other one thing towards forming habits of carefulness and accuracy.

30. Methods of Checking Addition. The following methods of checking addition are in common use.

1. By adding in the reverse order. If the sum is obtained by adding from the bottom up, it can be checked by adding again starting at the top. The two results should be the same. Reversing the order serves to give different combinations thus avoiding the tendency to repeat any mistake made in the original addition.

2. By adding again omitting the first addend, and adding it later to the sum obtained by adding the other addends.

Example:	685	Check:	3653
	747		856
	564		4509
	378		
	897		
	382		
	856		
	4509		

A convenient way to do this is to place a card over the bottom addend (856) and add the others, putting the sum (3653) on

the card. Then copy the omitted addend (856) on the card and add. The result so obtained should be the same as that obtained by adding the column in the usual way. Omitting one addend changes the combinations met with in the addition and so avoids the danger of repeating an error.

3. By casting out the nines. This method of checking is discussed in detail in the next section.

V. Checking by Casting Out the Nines

31. Definition. The *excess of nines* in any number is the remainder obtained when that number is divided by nine. Thus 33÷9 gives a quotient of 3 with a remainder or *excess* of 6.

32. How to Find the Excess. Evidently the excess of nines in any number may be obtained by dividing that number by 9, the remainder obtained being the excess. It happens, however, that there is a shorter way of obtaining the excess of 9's in a number than by dividing the number by 9. This short method is evident from the following cases:

33÷9=3 with an excess of 6
43÷9=4 with an excess of 7
17÷9=1 with an excess of 8

In each of these cases the excess of nines is equal to the sum of the digits.

48÷9=5 with an excess of 3
76÷9=8 with an excess of 4

The sum of the digits of 48 is 12 and the excess of nines in twelve is 3. The sum of the digits of 76 is 13 and the excess of nines in thirteen is 4. It is seen from the above that *the excess of nines in any number is equal to the excess of nines in the sum of its digits.* A proof of this principle is suggested in Section 61, Exercise 7.

Thus the excess of nine in 8756 is the same as the excess in the sum of 8+7+5+6 or 26. The easiest way to find the

excess in 26 is to apply the same principle again, namely, that the excess in 26 is the same as the excess in 2+6 or 8. That 8 is the true excess of nines in 8756 can easily be verified by dividing 8756 by 9. It follows that:

To find the excess of nines in any number add the digits, if the sum has two digits add again. Continue until a sum of one digit is obtained which will be the required excess unless the digit is nine when it should be thrown away, the excess being zero.

The process can be shortened by leaving out 9's and combinations of digits that make 9 in adding the digits. This process of finding the excess of nines in a number is called *casting out the nines.*

EXERCISE. CASTING OUT NINES

Find the excess of 9's in the following numbers:

1. 47596. Notice that one digit is 9 and that 4 and 5 make 9. Adding the other digits 13 is obtained. Adding 1 and 3 gives 4 the excess of 9's in 47596.

2. 28576. The sum of the digits is 28, 2+8=10. 1+0=1 the excess of nines in 28576.

3. 84762. Notice that 7 and 2 make 9, adding the others gives 18. 1+8=9. Throwing away the 9 the excess is 0.

4. 81478
5. 5697
6. 57394
7. 68973
8. 358972
9. 783484
10. 8527543

33. Checking Addition by Casting out the Nines. If two numbers are added and the excess of nines in the first is 5 and in the second 2, it is evident that the excess of nines in the sum will be 5+2 or 7. If the excess in one number is 8 and in the other 7 it is evident that their sum will not have an excess of 15, for 15 will contain another nine leaving an excess of 6. From this the truth of the following principle is seen.

Principle. *The excess of nines in the sum is equal to the excess in the sum of the excesses of the addends.*

Example:	875	The excess in 875 is	2
	546	The excess in 546 is	6
	734	The excess in 734 is	5
	657	The excess in 657 is	0
The sum is	2812	The sum of the excesses of the addends is	13
The excess in the sum is 4		The excess in the sum of excesses of the addends is....	4

Check

4 = 4

For convenience the work should be arranged as follows:

875	2
546	6
734	5
657	0
2812	13

4 = 4

34. Limitations. The check by casting out the nines does not afford an absolute proof of the correctness of the sum. The result will apparently "check" as long as the excess in the sum obtained is the same as the excess in the sum of the excesses of the addends. As many numbers have the same excess it follows that the sum obtained might be wrong but if it has the same excess as the correct sum the check of casting out nines would not detect the error. The correct sum in the above example is 2812 but 2821, 2722, and 1903 would all apparently check as they all have the same excess as the correct sum. The other two checks suggested are therefore to be preferred in addition as they are more reliable and not much longer than the check by casting out nines.

VI. Short Methods in Addition

35. Use of Short Methods in the Elementary School. There are not many short methods in addition that are suitable to teach in the elementary school. The regular method of adding should be thoroughly mastered and habituated before any short methods are taught. In the upper grades, however, many of the pupils can profit by suggestions for shortening the work. Short methods not only have a practical value but also serve to introduce something new into the drill on funda mentals in the upper grades and as a result make the work more interesting.

36. Useful Short Methods. The following short methods will be found useful.

I. Where a given digit is repeated several times in the same column the work of adding may be shortened by counting the number of times the digit is repeated and multiplying. Thus in finding the number of days from January 14 to September 24, 1921, we might proceed as follows:

January	17 days
February	28 days
March	31 days
April	30 days
May	31 days
June	30 days
July	31 days
August	31 days
Sept.	24 days
	253 days

In adding, first count the number of 1's in the first column (4) and then think 8, 16, 23. Carrying the 2 and adding to the 2 in the second column gives 4. Then counting the 3's gives $7\times3=21$ and adding the 4 gives 25.

II. Grouping. As soon as they are able to do so pupils should be encouraged to group small digits together and handle them in one addition instead of several.

Example:
9
4
3
8
6

In adding this column the pupils have been in the habit of thinking 14, 17, 21, 30. As soon as they are ready to do so without becoming confused they should be encouraged to group the 3 and 4 and think of it as 7. Then in adding they would think 14, 21, 30.

The digits should be grouped as they come. It is doubtful if anything is gained by skipping around so as to get digits that make 10, or other easy combinations.

While most pupils will probably resort to grouping in only the easiest cases some can be taught to group systematically always handling two or more figures at a time.

Example:
7
5
8
7
9
5

In adding this column look at 5 and 9 and see 14, look at 7 and 8, and see 15, look at 5 and 7, and see 12. In adding think only 14, 29, 41. Mastery of this method demands two things (1) the ability to recognize instantly the combinations and (2) the ability to add mentally two two-place numbers as 14 and 15, and 29 and 12.

Exercise. Checking and Short Methods.

1. On what principle of addition are the first two methods of checking, suggested in the text, based?

2. Add and check by the first method suggested. Practice grouping easy combinations.

a.	b.	c.	d.
127	996	237	386
375	320	949	463
953	778	486	827
333	886	987	240
325	913	354	616
911	164	600	261
554	897	744	755
167	972	195	833
554	119	234	959

3. Add and check by the second method suggested. Group easy combinations.

a.	b.	c.	d.
186	474	877	537
775	787	845	685
684	591	981	452
260	106	693	904
372	869	184	511
846	451	772	988
595	336	749	559
254	820	256	127
137	533	258	323

4. Add and check by casting out the nines. Group easy combinations.

a.	b.	c.	d.
237	564	632	674
492	278	263	158
679	947	318	745
513	522	949	121
468	989	746	437
731	243	653	426
856	334	428	953
302	669	456	674
925	142	532	329

CHAPTER III.

SUBTRACTION

I. The Meaning and Use of Subtraction

37. The Three Subtraction Ideas. There are three subtraction ideas or three types of subtraction situations with which the pupils in the primary grades must become familiar. By the end of the third school year the pupils should be able to recognize the three types of problems that are solved by means of subtraction. They can only obtain this ability by meeting plenty of concrete problems of all three types. The types are as follows:

(a) The Remainder or Take Away Type. "If Johnny has 7 pieces of candy and eats 2, how many does he have left?"

(b) The Difference or Comparison Type. "Mary has 7 apples and Julia has 2. How many more apples does Mary have than Julia?"

(c) The Additive Type. "Johnny sees a toy in a shop window marked 10¢. If he has 7¢ how many more must he have to buy the toy?"

No matter what the pupils think in performing the abstract process of subtraction* they must learn through experience that subtraction is used for three purposes (a) to find how much is left when a smaller number is taken from another, (b) to find how much larger or smaller one number is than another, and (c) to find how much must be added to a smaller number to make a larger.

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*See Section 41.

38. Definitions. Mathematically, *subtraction* is simply the inverse of addition. It is the process by which the second addend is found when the sum of two addends and one of the addends is known.

Other definitions that are often given for subtraction are as follows:

Subtraction is the process of taking a smaller number from a larger to find how much is left.

Subtraction is the process of comparing two numbers to find how much larger or smaller the one is than the other.

Subtraction is the process of finding how much must be added to a smaller number to make a larger.

The last three definitions of subtraction are only *partial* definitions each being based upon and including only one of the three ideas of subtraction. The three together would constitute a fairly complete definition of the process.

The word *subtract* comes from two Latin words *sub,* under, and *trahere,* to draw.

Latin arithmetics used the terms *numerus minuendus* (a number to be diminished), and *numerus subtrahendus* (a number to be subtracted). These expressions came into English as *minuend* and *subtrahend.*

Two names are in common use in the schools for the result in subtraction. The term *remainder* comes from the take away idea of subtraction while the other term *difference* comes from the comparison idea. In bookkeeping and in business in general the term *balance* is in common use.

II. Principles of Subtraction

39. Principles. (1) Only quantities having the same name (expressed in the same units) can be subtracted and the result always has the same name as the original numbers.

(2) The same quantity or equal quantities can be added to the minuend and subtrahend without changing the result.

Thus 8—3=5, adding 4 to both minuend and subtrahend gives 12—7 but the result is still 5. This principle is used in one method of subtracting.*

(3) The sum of the difference and subtrahend equals the minuend.

III. The Process of Subtraction

40. Old Methods of Subtracting. The process of subtraction, with the Hindus, like that of addition, started either at the left or the right, but usually at the left. This method of starting at the left is also found among Arab writers and in Europe as late as the end of the sixteenth century. The Hindus employed several different methods of subtracting.

The following example shows one method of subtracting in Europe during the sixteenth century. It is similar to one of the methods used by the Hindus, who however, erased the figures instead of crossing them out. The example shows the subtraction of 345 from 432, the result being 87.

87 Starting at the left 3 from 4 (crossing out the 3 and 4) leaves 1, but as the following subtrahend figure, 4, is larger than the minuend figure, 3, the 1 was not written. 4 from 13 (crossing out the 4 and 3) leaves 9, but as the next subtrahend figure 5 is greater than the minuend figure 3, 1 was taken from the 9 and the result, 8, written above the 3. 5 from 12 (crossing out the 5 and 2) leaves 7, which was written above the 2.

Exercise. Old Methods of Subtracting

Subtract the following using the method described above.

1.	2.	3.	4.
576	7135	8597	3476
189	2519	4738	1791

*See Section 41.

41. Present Methods of Subtracting. In subtracting when each figure in the subtrahend is smaller than the corresponding figure in the minuend, two methods of thinking are in common use.

7586
5135

In the example given one can think, beginning at the right, 5 from 6 leaves 1 (writing the 1), 3 from 8 leaves 5 (writing the 5), 1 from 5 leaves 4 (writing the 4), 5 from 7 leaves 2 (writing the 2). This is the *take away* method.

7586
5135

Or, in the same example, one can think 5 and 1 (writing the 1) make 6, 3 and 5 (writing the 5) make 8, 1 and 4 (writing the 4) make 5, 5 and 2 (writing the 2) make 7. This is the *additive* method.

The first of these methods of thinking is based on the take away idea of subtraction and the second on the additive idea. Theoretically, one is as much subtraction as the other, they are simply different ways of thinking in performing the abstract process of subtraction.

The chief difficulty that occurs in the process of subtraction is the difficulty of subtracting when the figure in the subtrahend is larger than the corresponding figure in the minuend. Several methods of overcoming this difficulty have been used at various times and two are in common use today. The following example should make these two methods clear to the student.

756
379

As 6, the units figure in the minuend, is smaller than 9, the units figure in the subtrahend, it is impossible to subtract the units as they are. To overcome this difficulty one of the 5 tens in the minuend is added to the 6 units making 16 units in the minuend. Subtracting the 9 units in the subtrahend gives 7 units in the result. As one of the 5 tens in the minuend has already been used there are now only 4 tens. Next 1 of the 7 hundreds is added to the 4 tens giving 14 tens in the minuend (as 1 hundred is the same as 10 tens). Subtracting the 7 tens in the subtrahend gives 7 tens in the result. As one of the 7 hundreds in the

minuend has already been used there are only 6 hundreds left. Subtracting the 3 hundreds in the subtrahend gives 3 hundreds in the answer. This method of overcoming the difficulty of larger subtrahend figures is usually called the *borrowing* method.

The second method is based on the principle that the same or equal quantities can be added to the minuend and subtrahend without changing the result. 6 units is smaller than 9 units so 10 units are added to the 6 units making 16 units in the minuend. Subtracting the 9 units in the subtrahend gives 7 units in the result. As 10 units have been added to the minuend an equal amount, namely 1 ten, is added to the subtrahend making 8 tens in the subtrahend. As 5 tens in the minuend is too small 10 tens are added making 15 tens. Subtracting the 8 tens in the subtrahend gives 7 tens in the result. As 10 tens have been added to the minuend an equal amount, namely 1 hundred, is added to the subtrahend giving 4 hundreds in the subtrahend. There are 7 hundreds in the minuend and 4 in the subtrahend giving 3 in the result. This method of overcoming the difficulty of the larger subtrahend figure is called the method of *equal additions* as it depends on adding equal quantities to both the minuend and subtrahend. The Hindus were familiar with and made use of both the borrowing and equal additions methods.

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 756
 379
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As either borrowing or equal additions can be combined with either the take away or additive methods of thinking there are altogether four different methods of subtracting being taught in America today.

I. Take Away-Borrowing.

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 4175
 2319
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In using this method one thinks:

9 from 15, 6
1 from 6, 5
3 from 11, 8
2 from 3, 1

II. Take Away-Equal Additions.

4175	9 from 15, 6
2319	2 from 7, 5
	3 from 11, 8
	3 from 4, 1

III. Additive-Equal Additions.

4175	9 and 6,	(Write the 6),	15
2319	2 and 5,	(Write the 5),	7
	3 and 8,	(Write the 8),	11
	3 and 1,	(Write the 1),	4

IV. Additive-Borrowing.

4175	9 and 6,	(Write the 6),	15
2319	1 and 5,	(Write the 5),	6
	3 and 8,	(Write the 8),	11
	2 and 1,	(Write the 1),	3

42. Comparative Merits of Different Methods. Need of Standardization. We have not as yet sufficient experimental evidence to warrant any definite conclusion as to which of the four methods given in Section 41 is the best. There is some evidence which seems to indicate that the Equal Additions plan of overcoming the difficulty of a larger subtrahend figure is somewhat more rapid and more accurate than the Borrowing plan. The prevalence of the latter plan is probably due to the fact that most teachers consider it somewhat easier to teach. There is no conclusive evidence as to whether it is better to teach the pupils to think according to the Additive or the Take Away plans. It is claimed by some people that the additive plan is more accurate and more rapid but this has never been proved. It is further claimed for the additive plan that it accomplishes a saving of time in the lower grades in mastering the facts or combinations. If the pupils think Take Away they must memorize not only the addition facts but also the subtraction facts. Corresponding

to the addition facts 7+8=15 and 8+7=15 they must learn the subtraction facts 15—8=7 and 15—7=8. On the other hand if they think according to the Additive plan they need no separate subtraction facts, if the addition facts are drilled upon in the forms 7+8=15, 8+7=15, 7+?=15 and 8+?=15. It is reasonable to suppose that this will result in a saving of time although this has never been proved or measured. Experience and what experimental evidence we have seem to indicate that the Take Away-Equal Additions and the Additive-Equal Additions are the two best methods with the Take Away-Borrowing third, but that the differences between these three methods are not very great. The Additive-Borrowing is very awkward and slow and should never be taught.

Which one of the first three methods mentioned above is taught is probably not very important. The method taught in our schools, however, should be standardized to avoid confusion and waste of time, and in order that our teachers may be trained to teach one method well.

43. An Analysis of the Subtraction Process with Suggestions for Teaching. The following outline gives the usual order of presenting the various points involved in the process of subtraction. Each point should first be understood and then reduced to habit through drill.

I. Facts. In order to be able to subtract any two numbers it is necessary that the pupils memorize the primary subtraction facts. These are the inverses of the addition facts. There are one hundred subtraction facts including those in which the answer is zero, as $\begin{array}{r} 0 \\ \hline 0 \end{array}$ $\begin{array}{r} 1 \\ \hline 0 \end{array}$ $\begin{array}{r} 2 \\ \hline 0 \end{array}$, etc., and those in which the subtrahend is zero, as $\begin{array}{r} 1 \\ \hline 1 \end{array}$ $\begin{array}{r} 2 \\ \hline 2 \end{array}$ $\begin{array}{r} 3 \\ \hline 3 \end{array}$, etc. The subtraction facts should

be taught at the same time as the addition facts and the two sets of facts should be drilled upon together. Such a procedure saves time because of the close relationship between the facts. Thus, 7, 8, 15 and 15 should be kept together in the drill

$$\underline{+8} \quad \underline{+7} \quad \underline{-7} \quad \underline{-8}$$

work. If the pupils think according to the Additive plan they think, in giving the fact 15 "7 and 8 make 15," whereas if

$$\underline{-7}$$

they think according to the Take Away plan they think "7 from 15 leaves 8."

If the Take Away method is to be taught the first subtraction facts should be worked out with objects. The later facts may be obtained from the addition facts. Thus the pupils can find the result of 12 the first time it is met by think-

$$\underline{-7}$$

ing that 12 is made up of 7 and 5, so if the 7 is taken away the 5 remains. If the Additive method is to be taught it is not necessary to use objects in working out the early subtraction facts as they are the same as the addition facts which have already been objectified. Thus the pupils find that 3 and 2 make 5 and drill on this relation in the form 3 and 3,

$$\underline{+2} \qquad \underline{?}$$
$$? \qquad 5$$

the last being the form used in subtracting. Once obtained the facts must be memorized through drill.

II. Two-place Numbers, Subtrahend Figures Smaller than Corresponding Minuend.

$$\begin{array}{rr} 98 & 75 \\ \underline{23} & \underline{42} \end{array}$$

As soon as a few subtraction facts are known they should be used in examples of this type. As in addition, in subtracting larger numbers they must be broken into their parts. The units are subtracted first and the result placed in units place, the tens next, etc.

III. Zero Difficulties.

56	28	30
36	10	10

The presence of a zero always gives more or less difficulty to beginners, so plenty of practice on examples such as the above must be provided. Some schools teach 30 or 3 tens at the same time as 3 or 3 units.

		10	1 ten
	1	1 unit	

IV. Two-place Numbers, Subtrahend Figure Larger than Minuend in Units Place.

53	40
18	23

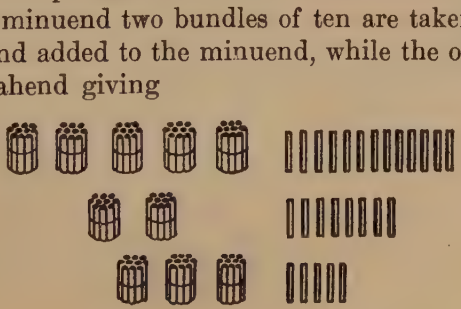
This is the biggest difficulty in the process of subtraction. An understanding of either of the methods used in overcoming this difficulty depends on a knowledge of place value. To make sure that pupils understand place value plenty of practice with the sticks, such as described in Section 28, must be given. The first subtraction examples involving this difficulty should also be worked out with the sticks. If the first case met is 53 18 the pupils should first represent the two numbers by the sticks as:

Then, if the Borrowing method is taught, they should be led to see that as they do not have enough units in the minuend they must take one of the tens and untie the bundle, placing the ten sticks with the units giving:

As soon as this has been made clear with the sticks they should try subtracting 53 without the sticks, thinking the same as $\begin{array}{r} 53 \\ -18 \end{array}$ when the sticks were used. Finally they should subtract thinking only what is necessary in order to get the result.

If the Equal Additions method is taught it may be developed in either of two ways. First from simple cases such as $\begin{array}{r} 7 \\ -3 \end{array}$ they should discover that adding the same number to both the minuend and subtrahend does not change the result. Thus adding 2 to both, the example $\begin{array}{r} 7 \\ -3 \end{array}$ is changed to $\begin{array}{r} 9 \\ -5 \end{array}$, but the result is still 4. Then with the sticks before them, as below,

the teacher can proceed as follows. As there are not enough units in the minuend two bundles of ten are taken, and one is broken up and added to the minuend, while the other is added to the subtrahend giving



As the same thing is added to the minuend and subtrahend the result is not changed.

Another way of developing that is often used when the Additive-Equal Additions method is taught, is as follows. From the review of simple cases such as 8 and 45 the pupils are led to see that the problem is always to find how much must be added to the subtrahend to make it equal to the minuend. Then with the sticks they start as before. The problem here is to find out what must be added to 18, or 1 ten and 8 units, to make it equal 53, or 5 tens and 3 units. Then starting with the 18, the teacher must help the pupils to see that by adding 5 units to the 8 units they will get 13 units or 1 ten and three units. This with the 1 ten they had to start with gives 2 tens and 3 units or 23. Then to get 5 tens they must add 3 more tens. So 5 units and 3 tens or 35 must be added to 18 to make 53. This should then be done without the sticks.

$$\begin{array}{r} 8 \\ \underline{3} \\ 45 \\ \underline{21} \end{array}$$

V. Three-place Numbers. (a) Subtrahend figures all less than corresponding minuend figures.

$$\begin{array}{r} 375 \\ \underline{123} \end{array}$$

(b) Subtrahend figure in units place only larger than minuend figure.

$$\begin{array}{r} 583 \\ \underline{256} \end{array}$$

(c) Subtrahend figure in tens place only larger than minuend figure, as 427. This should be developed, if necessary, with the sticks in the same way as suggested for two-place numbers.

$$\begin{array}{r} 427 \\ \underline{243} \end{array}$$

(d) Subtrahend figures larger than minuend in both units and tens places.

$$\begin{array}{r} 623 \\ \underline{248} \end{array}$$

VI. Zero Difficulty.

$$\begin{array}{r} 302 \\ \underline{123} \end{array}$$

Examples of this type give difficulty if the Borrowing plan is used as, since there is a zero in tens place, there is nothing

to borrow from until 1 hundred (or 10 tens) is borrowed from hundreds place. Plenty of practice must be given on this type. It will cause less difficulty if the Equal Additions plan is taught.

VII. Subtrahend With Fewer Places than Minuend. Enough examples of this type must be given to fix the habit of placing units under units, tens under tens, etc.

VIII. Subtraction of Larger Numbers Having Four or More Digits.

Exercise. Developing Borrowing and Equal-Additions.

1. Work out the following examples in detail with the sticks using the Borrowing plan.

45	728	403
27	342	159

2. Work out the same examples with the sticks according to the Equal Additions plan, using the first method described.

3. Work out the same examples with the sticks according to the Additive-Equal Additions method using the second method described.

IV. Checking Subtraction

44. Methods of Checking. The most satisfactory check in subtraction is to use the principle that the difference added to the subtrahend should give the minuend. The check should be performed mentally and its use should be made habitual. Having subtracted the pupil should mentally add the difference and subtrahend noticing whether each figure as it is obtained is the same as the corresponding figure in the minuend.

Exercise. Different Methods of Subtracting. Checking

1. Subtract the following, using the Take Away-Equal Additions method. Check all results.

114957187	94752808	106089449	99833978
90271797	67349640	16915390	73160227

2. Subtract the following, using the Additive-Equal Additions method. Check all results.

115171700	82484740	115916913	72229470
63087381	48207825	55536329	45049173

3. Subtract the following, using the Take Away-Borrowing method. Check all results.

146246252	80630266	124485018	107419373
52160891	68164329	73098624	65348405

CHAPTER IV.

MULTIPLICATION

I. The Meaning and Use of Multiplication

45. The Multiplication Idea. Pupils can obtain a clear idea of the meaning and use of multiplicaiton only through experience with simple multiplication situations. Thus in playing store they meet the situation "If apples cost 3¢ apiece, how much must you pay for 4 apples?" At first such situations are worked out objectively. Using toy money the pupils count out 3¢ for one apple, 3¢ for another, 3¢ for the third, and 3¢ for the fourth. By counting they find that four groups of 3¢ each make 12¢. They are told that this is called multiplication, written $4\times3¢=12¢$, and is read 4 times 3¢ is 12¢. Through experience of this kind the pupils learn that 4×3 means 4 groups of 3 each, or four 3's and that 7×8 means 7 groups of 8 each, or seven 8's. Later, instead of using objects, they find the total cost of 4 things at 3¢ apiece by putting down 3¢ for each and adding, as shown at left. They thus get the more abstract idea of multiplication, namely that $4\times3=3+3+3+3$, and $7\times8=8+8+8+8+8+8+8$.

3¢
3¢
3¢
3¢
12¢

46. Definitions. The usual elementary definition of multiplication is as follows:

Multiplication is a short process of finding the sum of a number of equal addends.

Our word *multiple* comes from the two Latin words *multus,* meaning many, and *plus,* one meaning of which was fold. Multiple therefore means literally many-fold. The process of

finding any multiple of a number thus came to be called *multiplication.*

Latin arithmetics spoke of the number to be multiplied, or taken many-fold, as the *numerus multiplicandus.* This came into English as *multiplicand.* The term *multiplier* we made by adding *er* to multiply just as we add *er* to the verb teach to signify the one who does the teaching. Today in America the multiplier is usually written first. Thus 3×4 is read 3 times 4 and means $4+4+4$.

Our term *product* simply means the result, the thing produced, just as farm products are the things produced on a farm.

Our word *factor* comes from the Latin *factus,* meaning to make. That is the factors of a number are literally, the things that make the number.

II. Principles of Multiplication

47. Principles. (1) The multiplier must be an abstract number. This principle results directly from the meaning of multiplication. 3×4 means 3 groups of 4 each. Since the multiplier tells the number of groups it must be an abstract number. $\$3\times4$ or 2 ft.$\times4$ ft. are meaningless and impossible expressions.

(2) The multiplicand may be either abstract or concrete. Thus 3×4 means $4+4+4$ and $3\times\$4$ means $\$4+\$4+\$4$ or three groups each containing $4.

(3) The product is abstract or concrete according as the multiplicand is abstract or concrete. If concrete, the product has the same name, or is expressed in the same units, as the multiplicand. Thus 3×5 apples $=$ 15 apples, that is 3 groups of 5 apples each make altogether 15 apples.

(4) In the case of abstract numbers the multiplier and multiplicand can be interchanged without changing the product. Thus $4\times7=7+7+7+7=28$ and $7\times4=4+4+4+4+4+4+4=28$.

This principle is made use of in arithmetic in many ways. In the first place, an early acquaintance with this principle enables the pupils to save time in memorizing the multiplication facts. If they know that $7\times8=8\times7$ and drill on both $7\times8=56$ and $8\times7=56$ the pupils can learn both facts almost as soon as they could learn either by itself.

This principle is also used to shorten the work of multiplication by enabling the calculator to choose the easiest multiplier. In thinking of and in stating the cost of 237 articles at \$2 each, the pupils should think and say $237\times\$2$ but in the actual multiplication the numbers should be considered as abstract and the result obtained by multiplying 237 by 2. In order to save time the pupils should form the habit of working with abstract numbers and labeling the results only.

III. The Process of Multiplication

48. Old Methods of Multiplying. The Hindus used several different methods of multiplying only one of which will be shown here. In multiplying 5×347, they wrote the multiplier to the right of the multiplicand. The work started at the left. First, $5\times3=15$, which was written above the multiplicand. At this stage the work appeared as shown below.

First Stage

15
347 5

Then $5\times4=20$. The 0 was written above the 4 and the 2 was added to the 5, which was erased and replaced with a 7.

Second Stage

170
347 5

Then $5\times7=35$. The 5 was written above the 7 and the 3 was added to the 0, which was erased and replaced with a 3.

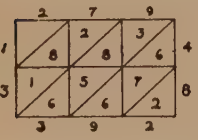
Final Stage
1735
347 5

A similar process was used when the multiplier contained more than one digit but it will not be given here as it is rather complex.

Among modern European nations the Italians have contributed most to the development of the art of calculation. From the fourteenth to seventeenth centuries they used several methods of multiplying. Only a few of these were original the others being obtained from the Hindus and Arabs.

I. This method is similar to our present method except that the successive partial products are not set over to the left but are added diagonally.

II. This method originated in Florence, Italy, sometime during the fourteenth or fifteenth century. The later Italian writers dropped the squares and it thus became the method we use today.

III.  The multiplication of 48×279 is here shown. The result is 13392. This method originated with the Arabs or Hindus and was at one time the most popular method throughout Europe. It was called the Latice method. This method is rather difficult to print so after the invention of printing, it was superceded by our present method.

Exercise. Old Methods of Multiplying

1. Multiply the following using the Hindu method described above.

1. 4×375 2. 3×346 3. 6×742

2. Multiply by the Latice method.

1. 34×576 2. 536×347

49. An Analysis of the Multiplication Process with Suggestions for Teaching. The usual order of teaching the different steps in the process of multiplication is given below together with suggestions for teaching. Each point should first be understood and then reduced to habit through drill.

I. The Multiplication Facts or Tables. In order to be able to multiply any two numbers by the present method it is necessary to memorize the products of any two numbers up to 10×10. There are altogether one hundred twenty-one of these facts. This number includes twenty-one combinations having 0 as one factor, $\underline{0}\ \ \underline{1}\ \ \underline{0}$, etc., nineteen others in which one factor is unity, as $\underline{1}\ \ \underline{2}\ \ 1$, etc., seventeen combinations in which one factor is ten, as $\underline{10}\ \ 2$, and twenty-eight reverses as $8\times 7=56$ the reverse of $7\times 8=56$. This leaves thirty-six of the more difficult combinations.

The first multiplications facts are developed through the counting of objects, as $3\times 4¢=12¢$, obtained by making three groups of $4¢$ each and counting the total number of cents; and by addition as $4\times 5=20$, obtained by adding four 5's as

$$\begin{array}{r} 5. \\ 5 \\ 5 \\ \underline{5} \\ 20 \end{array}$$

Later facts can be developed by counting as $7\times 8=56$, obtained by counting by 8's; or from facts already known as $8\times 7=56$, from $7\times 8=56$, or from $7\times 7=49$ by adding another 7. As soon as developed they must be memorized through drill.

The multiplication facts are taught in various orders. Some schools first present them in tables and present the tables in arithmetical order, as follows:

$1\times 1=1$ $2\times 1=2$
$1\times 2=2$ $2\times 2=4$
$1\times 3=3$ $2\times 3=6$
* * * * * *
$1\times 9=9$ $2\times 9=18$
$1\times 10=10$ $2\times 10=20$ etc.

Some schools prefer organizing the tables in the following form:

$1\times 1=1$ $1\times 2=2$
$2\times 1=2$ $2\times 2=4$
$3\times 1=3$ $3\times 2=6$
$4\times 1=4$ $4\times 2=8$ etc.

The table of 2's is usually taught first, as it brings out the multiplication idea better than the 1's, the table of 3's next, and so on in arithmetical order, the 1's being introduced later.

Some schools instead of teaching the tables in arithmetical order teach them in the order of difficulty so far as that is known. When this is done the 2's, 4's, 5's, and 10's are usually taught first in the order given and then the other tables are taught in various orders.

Some schools do not present the facts in tables but present them as individual facts in various orders. It probably makes little difference in what order they are first presented. The important thing is that the order be constantly changed in the drill work so that each fact will be learned as a separate fact and not as part of a table.

II. One-place Multiplier and Two-place Multiplicand, no carrying. As soon as a few facts are known they should be used in examples of this type. As in addition, when the numbers are too large to handle all at once they must be broken up into their parts and the parts handled separately. 3×23 is not known, but by breaking 23 into 2 tens and 3 units, we can think 3 times 3 units is 9 units, writing the 9 in units place, and 3 times 2 tens is 6 tens, writing the 6 in tens place.

$$\begin{array}{rr} 23 & 32 \\ \underline{3} & \underline{4} \end{array}$$

III. One-place Multiplier and Three-place Multiplicand, No carrying. 214
2

IV. Zeros in Multiplicand. 40 220 204 400. It took the
3 4 2 2
world a long time to see the need of a symbol for nothing. The pupils in the grades have the same difficulty. In 40 they have
3
a tendency to think $3\times0=0$ and since the product is 0 they do not put anything down getting 12 instead of 120 as the result. Another very common mistake is to say $3\times0=3$. Plenty of practice must be given on examples containing ciphers in the multiplicand. Some schools teach 40 and 400 at the same
3 3
time as 4.
3

V. Carrying. (a) From units to tens. 125. (b) From tens
3
to hundreds 252. (c) In both units and tens places 125.
3 5

As in addition the understanding of the process of carrying depends on the understanding of place value. If this has been well taught in connection with addition and subtraction it should give no great difficulty here. All that is usually necessary is to think through an example or two with the class. Thus in 125 we think first 3×5 units is 15 units or 5 units
3
and 1 ten. We write the 5 in units place and remember the 1 and add it to the tens. Then 3×2 tens is 6 tens, and 1 ten is 7 tens, etc. After they have gotten the idea by thinking through one or two examples in detail, the pupils should perform the work mechanically without thinking anything about units or tens. Even after carrying in multiplication is thoroughly understood the pupils may not perform it readily as

in order to do so they must be able to remember the number carried and add it to the result obtained in the next multiplication. To help them in this plenty of rapid fire oral drill should be given on examples such as $3\times9+2=?$, $4\times7+1=?$

VI. Zeros in Product. 125. The difficulty here is similar to
4
that in *IV* above.

VII. Multiplication by 10. As early as possible the pupils should discover and make use of the short method of multiplying by 10. This is usually discovered from the multiplication tables. The pupils know that $10\times2=20$, $10\times3=30$, $10\times7=70$, etc. and from this can generalize that to multiply any number by ten it is only necessary to annex a cipher.

The short method of multiplying by 10 might also be discovered by addition, as $10\times27=270$ which could be obtained by putting down ten 27's in a column and adding.

A third method is to take an example such as 237 and
10
multiply it out just as if the multiplier were 7 or 9 or any other one-place number. Thus $10\times7=70$, put down the zero in units place and carry the 7; $10\times3=30$, and 7 makes 37, put down the 7 and carry the 3; $10\times2=20$, and 3 makes 23; giving 2370 as the result. By any of these methods the pupils can discover for themselves the short way of multiplying by 10.

The explanation of the short method of multiplying by 10 is very simple, although it need not be attempted in the grades unless the pupils ask for an explanation. It is better for the pupils to discover the method than for the teacher to explain it. After the pupils have discovered the method the teacher should give the explanation if it is asked for or if it seems desirable. Annexing a cipher to the right of any number pushes each digit one place to the left, and so puts each digit into a place having a value 10 times that of the place originally occupied by the digit. Thus the value represented by each

digit, and therefore the value of the entire number, is multiplied by 10. Thus when 0 is annexed to 237 we have 2370, and the 7 is pushed from units into tens place, the 3 from tens into hundreds and the 2 from hundreds into thousands.

VIII. Multiplication by 20, 30, 40, etc. Since $20=2\times10$, to multiply by 20 is the same as to multiply by 10 and that result by 2. So to multiply by 20 we annex a cipher (multiply by 10) and multiply by 2. In practice the pupils should use the following form:

$$\begin{array}{r} 234 \\ 20 \\ \hline 2680 \end{array}$$

IX. Two-place Multipliers. (a) Without carrying. (b) With carrying. To multiply 232 by 23 means to find 23 times the multiplicand, 232. We cannot multiply by 23 all at once so we must break up the 23 into 20 and 3. If we find 3 times 232 and then find 20 times 232 and add we will evidently have 23×232, which is what we want. So to multiply by 23 we first multiply by 3 and then by 20 and add these results. When the pupils understand this they can put down the example

$$\begin{array}{r} 232 \\ 23 \\ \hline \end{array}$$

multiply by 3 in the usual way; and by 20, by bringing down a cipher and multiplying by 2, and finally add. The work then looks as follows:

$$\begin{array}{r} 232 \\ 23 \\ \hline 696 \\ 4640 \\ \hline 5336 \end{array}$$

The next step is to consider with the pupils what part of this work is necessary and what unnecessary. In the first place the 0 is not needed as it does not change the result, $0+6$ being just 6. So in multiplying by the 20 all that is necessary is to multiply by 2 and place the first figure in tens place. The pupils' final conclusion should be that to multiply by a two-place number they first multiply by the digit in units place and

place the right hand figure of the product in units place, then multiply by the digit in tens place and place the right hand digit of the product in tens place, and then add.

X. Multiplication by 100. Since $100=10\times10$ to multiply by 100 is the same as to multiply by 10 and that result again by 10. To multiply by 10 we annex a cipher and to multiply by 10 again we annex another cipher. So, to multiply by 100, we annex two ciphers.

XI. Multiplication by 200, 300, 400, etc. To multiply by 200 is the same as to multiply by 100 and then by 2, as $200=2\times100$. Therefore to multiply by 200 we annex two ciphers (multiply by 100) and multiply by 2. The work should be put down as follows:

$$\begin{array}{r} 342 \\ 200 \\ \hline 68400 \end{array}$$

XII. Three-place Multipliers. To multiply 213 by 232 is the same as multiplying by 2, by 30 and by 200 and adding the results. The pupils know how to do all of these so they can put down the work as follows:

$$\begin{array}{r} 213 \\ 232 \\ \hline 426 \\ 6390 \\ 42600 \\ \hline 49416 \end{array}$$

Again the ciphers do not change the result so in practice they are omitted. The pupils' final conclusion should be that to multiply by a three place number first multiply by the digit in units place, then by the digit in tens and then by the digit in hundreds place; placing the right hand digit of each product immediately under the digit being used as a multiplier. Finally the partial products must be added to obtain the complete product.

XIII. Zero in Multiplier. Whenever there is a final zero in the multiplier the same plan should be used as in multiplying by 20, or 30. Thus in multiplying 756 by 380 the work should appear as follows:

$$\begin{array}{r} 756 \\ 380 \\ \hline 60480 \\ 2268 \\ \hline 287280 \end{array}$$

As $380 = 38 \times 10$ multiplying by 380 is the same as multiplying by 10 (annexing a cipher) and by 38.

The case of an intermediate cipher in the multiplier will give little trouble if the pupils have formed the habit of always placing the right hand figure of each partial product immediately below the figure being used as a multiplier. Thus, in multiplying 342 by 308, they first multiply by 8 and put the right hand figure of the product under the 8, then by 3 and put the right hand figure of the product under the 3, and add. The work appears as follows:

$$\begin{array}{r} 342 \\ 308 \\ \hline 2736 \\ 1026 \\ \hline 105336 \end{array}$$

Plenty of drill must be provided on examples having ciphers in the multiplier.

EXERCISE—DEVELOPING THE PROCESS OF MULTIPLICATION

1. Work out in detail as you would in presenting it in the grades, the process of multiplying by a three-place number.

IV. CHECKING MULTIPLICATION

50. Methods of Checking. Three methods of checking the result in multiplication are in common use.

(1) Check by interchanging the multiplicand and multiplier. The result obtained should be the same as in the original multiplication.

Example:	382	Check:	534
	534		382
	1528		1068
	1146		4272
	1910		1602
	203988		203988

(2) Check by dividing the product by the multiplier. The quotient obtained should equal the multiplicand, and there should be no remainder.

Example:	382	Check:	382
	534		534)203988
	1528		1602
	1146		4378
	1910		4272
	203988		1068
			1068

Note: Dividing the product by the multiplicand does not give as good a check as dividing by the multiplier, as the multiplications involved in the division are the same as the original multiplications and any error is apt to be repeated in the check.

The two methods given above give effective checks but are both rather long. The shortest method of checking multiplication is by casting out the nines. The check by casting out the nines is always open to the objection that it sometimes fails to detect errors. It is so much shorter than any other method of checking multiplication and division that it should be used in these processes as it rarely fails to detect errors.

(3) Check by casting out the nines. The checking of multiplication by casting out the nines is based on the following principle.

Principle. The excess of nines in the product of two numbers is always equal to the excess in the product of the excesses of the two numbers.

Example: 382 The excess in 382 is..... 4
534 The excess in 534 is..... 3
1528 The product of the excess is.....12
1146
1910
203988

The excess in the product is 3. The excess in the product of the excesses is 3.

In practice the work may be arranged as below:

382 4
534 3
1528 12
1146
1910
203988
3 = 3

The principle stated above can be proved algebraically as follows. Any number is composed of some number of 9's plus some excess. For example $77=8\times9+5$. It follows that any number may be represented by an algebraic expression of the form $9M+E$, where M represents the number of 9's in the number and E the excess. Let $9M+E$ represent one of two numbers to be multiplied and $9M'+E'$ the other. Then we can multiply the two numbers algebraically.

$$\begin{array}{l} 9M + E \\ 9M' + E' \\ \hline 81\,MM' + 9\,M'E + 9\,ME' + EE' \end{array}$$

The first three terms of the product are divisible by 9 so the only excess or remainder that is obtained by dividing the product by 9 will be that obtained by dividing the term EE' by 9. But EE' is the product of the excesses. Therefore the excess of 9's in the product of two numbers equals the excess in the product of the excesses of the two numbers. No attempt should be made to prove this principle in the grades. It should be used without proof, its truth being first verified in a few simple cases.

In multiplying by a number of more than one digit, several multiplications, and finally an addition must be performed. Each of these should be checked as soon as finished, by going over them carefully a second time. In this way many errors may be detected at once. This habit together with checking the final result by casting out the nines, will enable the pupils to detect most of their errors in multiplication.

V. Short Methods in Multiplication

51. Useful Short Methods. The following types of short methods in multiplication should be taught in the grades.

(1) Short methods of multiplying by powers of ten, as 10, 100, 1000, etc., and by numbers ending in ciphers. Multiplication by 10 and 100 and by numbers ending in ciphers has already been discussed. From the cases already considered the pupils should generalize and conclude that to multiply by any number ending in one or more ciphers, we bring down the ciphers and multiply by the rest of the number in the usual way.

Example:

349		7
23000		5
1047000		35
698		
8027000		
8	=	8

(2) Short methods of multiplying by aliquot parts of 100. An aliquot part of a number is any unit fractional part of the number as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. The most commonly used aliquot parts of 100 are $50=\frac{1}{2}$ of 100, $33\frac{1}{3}=\frac{1}{3}$ of 100, $25=\frac{1}{4}$ of 100, $20=\frac{1}{5}$ of 100, $16\frac{2}{3}=\frac{1}{6}$ of 100, $12\frac{1}{2}=\frac{1}{8}$ of 100, $10=\frac{1}{10}$ of 100, and $8\frac{1}{3}=\frac{1}{12}$ of 100. All of these numbers are easy multipliers because of the simple relation in which they stand to 100 which is itself an easy multiplier. Thus to multi-

ply any number by $33\frac{1}{3}$ we annex two ciphers and divide by 3. This is the same as multiplying by $33\frac{1}{3}$ because annexing the two ciphers multiplies by 100 which is $3\times33\frac{1}{3}$. The work of multiplying $33\frac{1}{3}\times257$ should appear as follows:

$$\begin{array}{r} 8566\frac{2}{3} \\ 3\overline{)25700} \end{array}$$

The student should state and explain similar short methods of multiplying by 25, $16\frac{2}{3}$, $12\frac{1}{2}$, and $8\frac{1}{3}$.

(3) The third type of short method is of somewhat wider application than the other two. If one part of the multiplier is a multiple of another part it is often easier to obtain the partial product from a previous partial product instead of from the multiplicand.

Example 1. Explanation.

$$\begin{array}{r} 836 \\ 84 \\ \hline 3344 \\ 6688 \\ \hline 70224 \end{array}$$

The first partial product is obtained in the usual way by multiplying 836 by 4. We want for the second partial product 8×836 but we already know $4\times836=3344$ so the easiest way to get 8×836 is to multiply 3344 by 2.

Example 2. Explanation.

$$\begin{array}{r} 342 \\ 26 \\ \hline 684 \\ 2052 \\ \hline 8892 \end{array}$$

Notice that 6 is a multiple of 2. First multiply 342 by 2 being careful to place the right hand figure of the product under the 2. Then we want for the next partial product 6×342. Since we have $2\times342=684$ the easisest way to get 6×342 is to multiply 684 by 3, being careful to place the right hand figure of the product under the 6.

Example 3.

757
735
5299
26495
556395

Explanation.

Notice that 35 is a multiple of 7. First multiply by 7, placing the product under the 7. Ordinarily we would have to multiply by the 3 and 5 separately as we can not multiply by 35 but in this case we can get 35×757 at once as we have $7\times757=5299$. So we next multiply 5299 by 5 (as $35=5\times7$) and place the product under the right hand figure of the 35.

Example 4.

34875
72636
209250
1255500
2511000
2533180500

Explanation.

Notice that 36 is a multiple of 6 and 72 is a multiple of 36. Multiply by the 6 in hundreds place, placing the product in hundreds place. Then, since $36=6\times6$, multiply the first product by 6 placing result under the right hand figure of the 36. Then, since $72=2\times36$, multiply the second product by 2, placing the result under the right hand figure of the 72.

Exercise. Short Methods and Checking.

1. State and explain short methods of multiplying by 25, $16\frac{2}{3}$, $12\frac{1}{2}$ and $8\frac{1}{3}$.

2. On what principle is the first method of checking multiplication, described above, based?

3. Multiply 375 by 297. Check by dividing the product by the multiplier 297. Also by dividing the product by the multiplicand 375. Which gives the best check? Why?

4. Multiply the short way.

1. $16\frac{2}{3}\times837$
2. 25×7569
3. $33\frac{1}{3}\times4576$
4. $12\frac{1}{2}\times591$
5. $33\frac{1}{3}\times871$
6. $8\frac{1}{3}\times5384$
7. $70576\times16\frac{2}{3}$
8. 8571×25

5. Give the results in the following without using pencil or paper.

1. $33\frac{1}{3} \times 27$
2. $24 \times 12\frac{1}{2}$
3. 25×19
4. $8\frac{1}{3} \times 36$
5. $87 \times 16\frac{2}{3}$
6. 50×432
7. $7 \times 33\frac{1}{3}$
8. $12\frac{1}{2} \times 17$
9. $9 \times 16\frac{2}{3}$
10. $31 \times 8\frac{1}{3}$

6. Multiply by the shortest way you can find and check by casting out the 9's.

1. 8259×28
2. 83×3467
3. 5289×39
4. 6473×740
5. 206×8632
6. 82×4637
7. 4926×620
8. 9624×903
9. 52000×847
10. 84×79
11. 612×757
12. 956×8700
13. 839×749
14. 32824×81764
15. 98749×57864
16. 57642×81900
17. 85764×45535
18. 98376×72836

CHAPTER V.

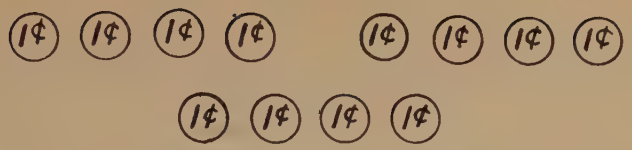
DIVISION

I. The Meaning and Use of Division

52. The Two Division Ideas. There are two division ideas or two types of division situations with which the pupils in the grades must become familiar. By the end of the third school year the pupils should be able to recognize the two types of problems that are solved by means of division. In order to do this they must meet plenty of concrete problems of each of the two types.

Division is the inverse of multiplication. If the product of two factors and one of the factors are known, the other is found by division. Corresponding to any multiplication situation there are two inverse division situations according as the known factor is the multiplier or the multiplicand. The multiplication statement $3 \times 4¢ = 12¢$ is a shorthand way of stating that 3 groups of $4¢$ each make $12¢$. The 3 tells the number of groups, the $4¢$ tells the number of cents in each group, and the $12¢$ tells the total number of cents. Corresponding to this multiplication situation we have the two division stituation $12¢ \div 3$ and $12¢ \div 4¢$.

In the division situation $12¢ \div 3$ we know the total number of cents ($12¢$) and the number of equal groups (3) and want to find the number of cents in each group. The problem, therefore, is to divide the $12¢$ into three equal groups and to find the number of cents in each group. This can be worked out objectively by taking 12 cents in toy-money and dividing it into 3 equal groups by putting 1 cent in each of three different places, then putting another with each of these, and another, etc. until all of the 12 cents are used. We would then have



The problem is to determine the number of cents in each group, so to find the answer we must count the number of cents in each group, the answer being 4¢. Since this type of division situation calls for the division or partition of a whole into equal parts it is often called *partition* or *partitive division*. The problem proposed by the partitive type of division is always that of dividing a whole (the dividend) into some number of equal parts (the number of parts being indicated by the divisor) in order to find the size of or the number in each part. A division situation of the partitive type can always be stated as a problem in fractions. Thus 12¢÷3, means the same thing as ⅓ of 12¢.

In the division situation 12¢÷4¢ we know the total number of cents (12¢) and the number of cents in each group (4¢). We are to find the other factor, namely the number of groups. The problem, therefore, is to find how many groups of 4 cents each can be made out of 12 cents. To work this out objectively, take 12 cents and place 4 cents in one group, 4 cents in another group, etc. until all of the 12 cents are used. The problem is to determine the number of groups, so to obtain the answer we must count the number of groups, the answer being 3. The problem in this type of division is always that of finding how many groups each equal to the divisor, can be formed from the dividend; or, briefly, how many times the divisor is contained in the dividend. Because of its similarity to the measuring process, in which we find how many times some unit is contained in the quantity being measured, this type of division is often called *measurement* or *measuring division*. A division problem of the measuring type can not be expressed as a problem in fractions.

Pupils must be given plenty of experience with both types of situations. They must have practice in finding how many times one number is contained in another. At first they should use objects, for example in finding how many 3's in 12, they can divide 12 into groups of 3 each and count the groups. Once obtained these results must be memorized. The term division is connected from the beginning with this problem of finding how many times one number is contained in another and the pupils are taught to write the problem in the form $12 \div 3$, or $3\overline{)12}$. The pupils' first idea of division is, therefore, that division is the process of finding how many times one number is contained in another.

The pupils must also have plenty of practice in finding parts of numbers. At first this type of work is not called division but is considered as work in fractions. The pupils in the primary grades handle such cases as $\frac{1}{2}$ of 12, and $\frac{1}{3}$ of 15 by using objects. As soon as the objective stage is passed the work of finding parts of numbers is connected with the work in division and the pupils are shown that the easiest way to find a part of a number is by division. Thus $\frac{1}{3}$ of 63 can be found by dividing 63 by 3. From this time on the pupils must use division in both types of situations and by so doing learn that division is used for two purposes (1) to find how many times one number is contained in another, and (2) to find parts of numbers.

Exercise—Two Types of Division

1. State what the problem or question is in each of the following and name the type of division.

(a) $\$24 \div 4 = ?$

(b) $\$24 \div \$4 = ?$

2. Work out the two situations given under Exercise 1 objectively, using toy money or some objects to represent dollars.

3. Show that 42÷6 can be thought of either as a partitive or as a measuring problem.

4. Show that 24÷$4 is meaningless, that is, that it is neither partitive or measuring division.

53. Definitions. The most general definition of division is that given in the preceding section, namely, *Division* is the inverse of multiplication. It is the process of findng the other of two factors, when the product and one factor are known.

Our word *divide* comes from the Latin word *dividere* meaning to part, or to separate.

Latin arithmetics used the terms *numĕrus dividendus,* meaning the number to be divided, and *numĕrus divisor* meaning the number by which you divide. From these expressions are derived our terms *dividend* and *divisor.*

Quotient comes from the Latin *quotus* meaning how often or how many times.

II. Principles of Division

54. Principles. (1) If the dividend and divisor are both concrete they must have the same name (be expressed in the same units). In this case the quotient is abstract and tells how many times the divisor is contained in the dividend. Examples: $12÷3 ft. is impossible. $12÷$3=4.

(2) If the dividend is concrete and the divisor abstract, the quotient is concrete and has the same name (is expressed in the same units) as the dividend. In this case the quotient represents the size of each of the equal parts into which the dividend has been divided. Example: 27¢÷3=9¢.

Pupils must be given plenty of practice in giving instantly the results in cases such as:

$45÷5=? 72¢÷8¢=?
$45÷$5=? 72¢÷8=?

(3) If both the dividend and divisor are abstract the quotient is also abstract and may be thought of either as telling the number of times the divisor is contained into the dividend, or as giving the size of the equal parts into which the dividend has been divided. Example: $12 \div 3 = 4$.

(4) The dividend and divisor may both be multiplied by the same number without changing the quotient. Example: $8 \div 2 = 4$. If 8 and 2 are both multiplied by 3 the example becomes $24 \div 6$ but the quotient is still 4.*

(5) The quotient times the divisor plus the remainder, if any, always equals the dividend. Example: 7 divided by 2 gives a quotient of 3 and a remainder of 1. $3 \times 2 + 1 = 7$.

III. The Process of Division

55. Old Methods of Dividing. So far as we know the Hindus used only one method of performing long division. It is of special interest as it was the method commonly used in Europe from the twelfth to seventeenth centuries and was not entirely extinct until late in the eighteenth century when it was replaced by our present method. As each figure was used the Hindus, working on the sand table, erased it so that when they had finished nothing remained but the result. In Europe the figures were crossed out or "scratched" instead of being erased. The division of 1728 by 12 is shown as it was performed in Europe. The divisor was written under the dividend and the quotient to the right as

1728 (Quotient)
12

*This principle is used in the division of decimals. See Section 100.

First Step.	*Explanation*
5 12|28(1 12	12 into 17 goes 1 time, which is written as the first figure of the quotient. 1×1 (cross out the 1 of the 12)=1; 1 from 1 (cross out the first figure of the dividend)=0; 1×2 (cross out the 2)=2; 2 from 7 (cross out the 7)=5; write the 5 above the 7.

Second Step	*Explanation*
1 12|28(14 12 1	Rewrite the divisor one place to the right. 12 into 52 goes 4 times. Write the 4 as the next figure of the quotient. 4×1 (cross out the 1) =4; 4 from 5 (cross out the 5)=1; write the 1 above the 5; 4×2 (cross out the 2)=8; 8 from 12 (cross out the 12)=4; write the 4 above the 2.

Third Step	*Explanation*
1 12|28(14 12 11	Rewrite the divisor one place to the right; 12 into 48 goes 4 times; write the 4 as the next figure of the quotient; 4×1 (cross out the 1)= 4; 4 from 4 (cross out the 4)=0; 4×2 (cross out the 2)=8; 8 from 8 (cross out the 8)=0.

This method was known as the Scratch Method or as the Galley Method. The last name was given to it because of the fancied resemblance of the work when completed to a galley, or large boat of the time. The galley method dropped out of use after the invention of printing because of the difficulty of printing it.

Our present method originated in Italy at least as early as the beginning of the fourteenth century.

Exercise. Old Methods of Dividing

1. Divide the following by the galley method.

1.	376÷2	3.	7384÷32
2.	8754÷9	4.	8597÷27

56. Present Methods of Dividing. The form for the work in long division has not been entirely standardized in America. All of the following forms have been taught in the schools within recent years.

Form I	*Form II*	*Form III*
212	37)7865(212	7865|37
37)7865	74	74 |212
74	46	46
46	37	37
37	95	95
95	74	74
74	21	21
21		

The first of these forms should be taught in all schools as it possesses several advantages, the most important being that it makes possible a more satisfactory method of dividing decimals. (See Sections 97 and 100.)

As the quotient is now usually placed above the dividend in long division there is a tendency to do the same in short division, as

$$8\overline{)7568}\quad 946$$

A shorter method than that given above is sometimes taught for long division, in which the multiplication and subtraction are performed at the same time, mentally, and only the remainder written. To do this the Additive-Equal Additions method of subtracting must be used. The method is best explained by means of an example.

First Step.	*Explanation.*
7 34)26671 28	In using the usual method the 34 is multiplied by the 7 and the product 238 placed under the 266. Then the 238 is subtracted from 266 giving a remainder of 28. In using the short method the multiplication and subtraction are per-

formed together and only the remainder written. This is done as follows: $7\times4=28$; *and* 8 (writing the 8) makes 36. $7\times3=21$; adding the 3 of the 36 gives 24; *and* 2 (writing the 2) makes 26. The *and* marks the subtraction by the additive method.

Second Step.	*Explanation.*
78 34)26671 287 15	Bring down the next figure. The next quotient figure is 8. $8\times4=32$; *and* 5 (writing the 5) $=37$. $8\times3=24$; adding the 3 of the 37 gives 27; *and* 1 (writing the 1) makes 28.

Third Step.	*Explanation.*
784 34)26671 287 151 15	Bring down the next figure. The next quotient figure is 4. $4\times4=16$; *and* 5 (writing the 5) $=21$. $4\times3=12$; adding the 2 of the 21 gives 14; *and* 1 (writing the 1) makes 15.

Just as it is possible to subtract by thinking in terms of addition, so it is possible to divide by thinking in terms of multiplication. Thus in the example $3\overline{)2576}$, instead of thinking 3 into 25, 8 and 1 over, we could think 3×8 (writing the 8) and 1 over; 3×5 (writing the 5) and 2 over; 3×8 (writing the 8) and 2 remainder. Although this method of dividing has been taught in this country it has never become as popular as the method of subtracting by adding.

57. An Analysis of the Division Process with Suggestions for Teaching. The usual order of teaching the various steps in the process of division is given below together with suggestions for teaching. Each point should first be understood and then reduced to habit through drill.

I. The even division facts as $2\overline{)4}$, $3\overline{)12}$. There are altogether ninety-nine of these facts if the multiplication facts are memorized to 10×10. This number includes nine facts in which the dividend is zero as $2\overline{)0}$, nine having unity as the quotient as $3\overline{)3}$, and nine of the form $10\overline{)20}$, $10\overline{)30}$, etc. The

division facts are learned at the same time as the inverse multiplication facts and the drill on the division facts is carried along side by side with the drill on the corresponding multiplication facts. Thus the four facts $3\times4=12$, $4\times3=12$, $4\overline{)12}^{3}$ and $3\overline{)12}^{4}$ are drilled upon together. The facts should be drilled upon in the partitive form ¼ of 12 as well as in the measuring form $4\overline{)12}$.

The first division facts are worked out objectively as described in the previous section. Later division facts are obtained from the multiplication facts. Thus the multiplication fact $7\times8=56$ states that seven 8's make 56, and therefore supplies the answer to the division question $8\overline{)56}$ which asks "How many 8's in 56?"

II. Short division, no carrying: $3\overline{)69}$, $3\overline{)129}$. Examples of this type should be taken up as soon as enough facts are known. In first thinking out such examples it is easier to think in terms of partitive division. Thus in $3\overline{)69}$ the pupils can think, 6 tens divided by 3 will be 2 tens, so we have two in tens place in the quotient, 9 units divided by 3 gives 3 units in the quotient. In $3\overline{)129}$ the first step is to divide the 12 tens by 3 which gives 4 tens, then 9 units divided by 3 gives 3 units. The result should be put above, or below if the teacher prefers, and each figure in its proper place, in tens place if it is tens and in units place if it is units. If put in its proper place the first figure of the quotient will always be above, or below, the last figure of the dividend used in the first division.

Although it is easier to reason out these examples in terms of partitive division it is probably better to think in terms of measuring in performing the division mechanically. So the pupils after thinking out a few cases should in practice think, in the example $3\overline{)129}$, "3 in 12, 4" and write the 4 above the 2 of the 12; "3 in 9, 3" and write the 3 above the 9.

III. Uneven division facts. As a preparation for carrying in division the pupils must be drilled upon situations such as $3\overline{)17}$, and $7\overline{)40}$ giving quotient and remainder. These facts are closely related to and derived from the even division and multiplication facts and the pupils already know the relations involved. Thus in $3\overline{)17}$ the pupils know that five 3's make 15, or that 3 goes into 15, 5 times. So it is evident that 3 goes into 17, 5 times with a remainder of 2. The pupils must be given plenty of drill on such situations so that they can handle them mechanically without having to stop and reason them out each time from other known facts.

IV. Short division with carrying. (a) One to carry, no remainder. $3\overline{)78}$. (b) Two or more to carry, no remainder. $6\overline{)84}$, $5\overline{)185}$. (c) Carrying in two places, no remainder. $2\overline{)734}$. (d) Final remainder. $3\overline{)68}$, $2\overline{)357}$. In first thinking out the example $3\overline{)78}$ the pupils think "7 tens divided by 3 gives 2 tens and 1 ten over. 1 ten and 8 units make 18 units divided by 3 gives 6 units." In practice they think "3 in 7, 2 and 1 over" and write the 2 above the 7; "3 in 18, 6" and write the 6 above the 8.

In the case of the final remainder it is perhaps best at first to leave it as a remainder or a part of the dividend that has not been divided, but that was left over when the rest of the dividend was divided. Thus in $3\overline{)68}$, 68 divided by 3 gives 3 parts of 22 each and 2 left undivided, that is, the quotient is 22 and the remainder 2. Later the pupils should go ahead and divide the remainder getting a fraction. In the case above if 68 is divided into 3 equal parts there are 22 in each part and 2 over. If the 2 is divided into 3 equal parts each part is $\frac{2}{3}$ of a whole, so if the division of 68 into 3 equal parts is completed there are $22\frac{2}{3}$ in each part, that is $3\overline{)68}$ $22\frac{2}{3}$

V. Zero difficulties. $4\overline{)840}$, $2\overline{)406}$, $4\overline{)816}$. Plenty of practice must be given on examples of these types.

VI. Long division, two-place divisors, the first figure of the divisor being used as a trial divisor and giving the correct quotient figure. In many respects long division is easier than short division. In short division the work is all done mentally and nothing but the result is recorded, whereas, in long division all of the work is written out in full. There is one difficulty, however, in long division which is not present in short division, namely, the difficulty of determining the quotient figure. The easiest type of example in long division is the type in which the correct quotient figure can be found by using the first figure of the divisor as a trial divisor. This type should be thoroughly mastered before the more difficult types are introduced.

As the process of long division is identical with the process of short division except that all of the steps are written down, the best introduction to long division is a review of short division. If sufficient drill has been given on short division the process has become mechanical, that is, the pupils perform the process without being conscious of the steps by which they reach the result. As a preparation for long division the pupils, by review, must be made conscious of the steps in short division. If the teacher desires, all of the work involved in a short division example, as $3\overline{)2874}$, may be written out instead of being performed mentally. Thus 3 goes into 28, 9 times; $9\times3=27$ so we have 1 left; 3 goes into 17, 5 times, $5\times3=15$ so we have 2 left; 3 goes into 24, 8 times. If all of these steps, which are performed unconsciously in short division, are written down we get

$$\begin{array}{r} 958 \\ 3\overline{)2874} \\ 27 \\ \hline 17 \\ 15 \\ \hline 24 \\ 24 \\ \hline \end{array}$$

which is the *form* used in long division.

After the steps in division have been made conscious and with an example such as the above before them, the pupils are ready for an example such as $21\overline{)672}$. As in short division the first question is "How many times does 21 go into 67?" If this gives difficulty the teacher can suggest that 21 goes into 67 about the same number of times as 20 into 60, or as 2 into 6, namely 3. Then $3\times21=63$, leaving 4. As in short division the 4 is used with the next figure of the dividend giving 42. Then 21 goes into 42, 2 times (trial 2 into 4); $2\times21=42$ leaving no remainder. The work when completed is as follows:

$$\begin{array}{r} 32 \\ 21\overline{)672} \\ 63 \\ \hline 42 \\ 42 \\ \hline \end{array}$$

After working out several examples the pupils should make a summary of the essential steps, something as follows:

1. Find the quotient figure by trying the first figure of the divisor into the first figure or first two figures of the dividend.
2. Multiply the divisor by the quotient figure.
3. Subtract.
4. Bring down next figure of dividend.
5. Find next quotient figure by trying first figure of divisor into this remainder, etc., repeating the same steps until all of the figures of the dividend have been used.

VII. Long division, two-place divisors, the first figure of the divisor being used as the trial divisor, but the true quotient figure one less than the trial quotient. Using the first figure of the divisor as a trial divisor will not always give the correct quotient figure. If the divisor is 43 and we use 4 as a trial divisor to obtain a trial quotient, we are really considering the divisor to be 40, whereas, it is larger than this, namely 43.

As our trial divisor is too small our trial quotient may be too large. The pupils must learn how to tell when the trial quotient is too large and how to correct the error. In the example $43\overline{)2924}$, using 4 as a trial divisor we get 7 as a trial quotient. Using this we get:

$$\begin{array}{r} 7 \\ 43\overline{)2974} \\ 301 \end{array}$$

Pupils often in a case like this subtract 1 from 7 and 0 from 9, getting 96, bring down the 4 and continue the division without noticing anything wrong. To prevent such errors the pupils must form the habit of comparing the product with the part of the dividend just above it and if the product is too large they must learn to locate the difficulty, namely, that the last quotient figure is too large. Much time and erasing will be saved if the pupils are encouraged to perform the multiplication mentally in order to see if the product is going to be too large.

VIII. Long division, two-place divisors, the trial divisor being one more than the first figure of the divisor and the trial quotient being the true quotient. An exceptionally weak pupil may find it easier to always use the first figure of the divisor as a trial divisor. The first divisor figure is not always, however, the best trial divisor, and the average pupil should know this, and not use the first figure in all cases. If the divisor is 21, 22 or 23 it is between 20 and 30 but nearer 20 than 30, so the nearest easy divisor is 20 or 2. If the divisor is 27, 28, or 29, it is still between 20 and 30 but nearer 30 than 20, so in these cases 30 or 3 is the nearest easy divisor. That is, if the second figure of the divisor is small, as in 32, the best trial divisor is the first figure of the divisor, in this case 3; but if the second figure is large, as in 48, the best trial divisor is 5 or one more than the first figure of the divisor.

The first examples in which the pupils use one more than the first divisor figure as a trial divisor should be carefully

chosen so that the trial division will give the true quotient figure in each case, as in the example.

```
    43          Trials (Mental):
57)2461                          4
   228          First trial    6)24
   181
   171          Second trial   6)18
    10
```

IX. Long division, two-place divisors, the trial divisor one greater than the first figure of the divisor and the true quotient one more than the trial quotient. If the divisor is 57 and 6 is used as a trial divisor, the divisor is really being regarded as 60 which is too large as it is really only 57. As the trial divisor is too large the trial quotient may be too small, as in the following example.

```
    2       Trial (Mental):
57)1767                  2
  114                  6)17
   62
```

To detect this mistake the pupils must form the habit of comparing the remainder to the divisor. If the remainder is larger than the divisor it is evident that the divisor would have gone into the dividend another time, that is the last quotient figure is too small and must be made one larger. Again the pupils should be encouraged to perform the multiplication and subtraction mentally before writing anything down so as to save time and erasing.

X. Long division, two-place divisors, quotient figure apparently one, but the partial product too large. 34)3230. In a case such as this, if the pupil tries 3 into 3 he gets 1 as the quotient figure but continuing he gets 34 as a product which is too large.

```
   1
34)3230
  34
```

In such a case the pupils should see that 34 will not go into 32 so they must use the next figure of the dividend making it 323. 34 goes into 323 almost 10 times, so the best trial quotient is 9. If this proves too large 8 should be tried next.

$$\begin{array}{r} 95 \\ 34\overline{)3230} \\ 306 \\ \hline 170 \\ 170 \\ \hline \end{array}$$

XI. Long division. Zero in quotient.

$$\begin{array}{r} 3 \\ 27\overline{)8235} \\ 81 \\ \hline 13 \end{array}$$

When, as in this example, the next figure of the dividend is brought down and the resulting number is less than the divisor, the pupils must learn to put a 0 in the quotient, to bring down the next dividend figure and continue the division.

XII. Long division. Three or more figures in divisor. As in the case of two-place divisors the pupils should use the first figure of the divisor, or one more than this figure as a trial divisor, according as the next figure is large or small. Before writing down the product they should test the trial quotient by mentally multiplying the first two figures of the divisor by the trial quotient. If the third figure of the divisor is large the second figure should be increased by one in the verification.

$$\begin{array}{rl} & 6 \\ 6 & \text{Trial (mental) } 4\overline{)25} \\ 379\overline{)25586} & \text{Verification (mental) } 6\times 8=48,\ 6\times 3=18, \\ & 18+4 \text{ (carried from the 48)}=22. \end{array}$$

EXERCISE. DEVELOPING THE PROCESS OF DIVISION

1. Using the following examples show that the first quotient figure always comes above the last figure of the dividend that is used in the first division.

1. $3\overline{)69}$ 2. $5\overline{)175}$ 3. $7\overline{)5286}$ 4. $21\overline{)4875}$
5. $32\overline{)1784}$

2. Work out in detail the process of carrying in short division.

3. Develop in detail the process of long division. Easiest case.

IV. Checking Division

58. Methods of Checking. (1) Divisor $\times$ quotient $+$ remainder$=$dividend.

Example	*Check*
43	43
$57\overline{)2461}$	57
228	301
181	215
171	10
10	2461

In using this check the divisor should always be used as the multiplier, for if the quotient is used as the multiplier the multiplications in the check will be the same as those in the original division and any error made in the multiplications is apt to be repeated. This is the best check to use in short division but in long division it is rather long.

(2) By casting out the nines. In checking division by excess of nines the same relation is made use of as in the check just given, but instead of using the divisor, quotient, remainder and dividend themselves, the excesses in these numbers are used. The method of checking is as follows.

Multiply the excess in the quotient by the excess in the divisor, add the excess in the remainder to this product, and find the excess in this result. This excess should be the same as the excess in the dividend.

Example	*Check*
7	$3\times7+1=22$ Ex. 4
3 43	4 (excess in 22)=4 (excess in dividend)
57)2462 —4	
228	
181	
171	
10 —1	

The method of checking division by 9's follows directly from the principles used in checking addition and multiplication. By the principle used in multiplication the excess of nines in the product of the divisor and quotient is equal to the excess in the product of the excesses of these two numbers. By the principle used in addition the excess in the sum of the product of the divisor and quotient and the remainder is equal to the excess in the sum of their excesses. But the sum of the product of the divisor and quotient and the remainder is equal to the dividend. So it follows that the excess in this sum must equal the excess in the dividend.

Because of its shortness the check by excess of 9's is the most convenient to use in long division and it will serve to detect most errors.

Besides checking the final result the pupils should form the habit of checking each step of the division as it is performed by going over it carefully a second time. If each multiplication and subtraction is verified in this way most of the errors will be detected at the time they are made and before the next step is performed.

V. Short Methods in Division

59. Useful Short Methods. The only short methods that should be taught in connection with the division of integers are the short methods of dividing by powers of ten. These are

often postponed until the work with decimals but should be taught for integers.

1. The pupils first learn to divide numbers ending in a cipher by 10, by dropping the final cipher. They can discover this by actual division as, $10\overline{)2830}$ 283. The result, 283, is the 2830 with the final cipher dropped. Dropping the cipher changes the 3 tens into units, the 8 hundreds into 8 tens and the 2 thousands into 2 hundreds and therefore divides the number by 10. In a similar way the pupils should discover the short way of dividing by 100, numbers ending in two or more ciphers, and by 1000, numbers ending in three or more ciphers.

2. The general case of dividing any number by 10, 100 or 1000 should also be taught in connection with integers. By actual division the pupils should discover that if any number is divided by 10 the right hand digit will be the remainder and the number formed by the other digits will be the quotient. Thus $378 \div 10 = 37$ with a remainder of 8. This is first discovered by division as: $10\overline{)378}$ 37 Remainder 8. In a similar way pupils should discover by dividing that when any number is divided by 100 the number formed by the two right hand digits is the remainder and the number formed by the other digits is the quotient. The similar short method of dividing by 1000 should also be taught.

Exercise. Checking and Short Methods.

1. By dividing several numbers, such as 3756, 41389, etc., by 1000, derive the short method.

2. Explain why, when any number is divided by 100, the number formed by the last two digits is the remainder, and the number formed by the other digits is the quotient.

3. State the quotient and remainder when each of the following numbers is divided by 10. By 100. By 1000.

1. 3879	3. 7890	5. 2700	7. 34000
2. 52476	4. 1392	6. 31875	8. 71500

4. Divide and check by casting out the 9's. Be careful to place the first quotient figure above the last figure of the dividend in the first division.

1. 716,431÷79	6. 731,876÷243
2. 98,191÷38	7. 2,819,757÷879
3. 346,271÷86	8. 547,863÷2,392
4. 87,452÷97	9. 1,807,560÷4,756
5. 217,854÷379	10. 8,756,071÷2,759

5. Divide the following using the short method in which the multiplication and subtraction are performed at the same time. Check by casting out the 9's.

1. 8756÷21	3. 56,876÷234
2. 24573÷38	4. 275,089÷835

VI. Tests of Divisibility

60. Teaching the Tests of Divisibility in the Grades. It is often necessary to know by what small numbers such as 2, 3, etc. a given number is evenly divisible. This is particularly important in reducing fractions to their lowest terms and in cancellation. As a minimum the pupils in the grades should be taught the tests for divisibility by 2, 3, 5, 9, and 10. The tests for 4 and 8 are useful but not indispensable. To test for 6 one must test for both 2 and 3 as $6=2\times3$. There is no known test for 7 that is shorter than actually dividing by 7 to see if the division is even.

Test for 2. Any number is divisible by 2 if the right hand digit is divisible by 2.

By division the pupils can see that 24, 86, and 138 are all evenly divisible by 2, but that 25, 87 and 139 are not. From these and similar cases the pupils can easily generalize and discover the test as stated above.

Test for 3. Any number is divisible by 3 if the sum of its digits is divisible by 3. Thus 258 is divisible by 3 because the sum of its digits, 15, is divisible by 3; but 347 is not divisible by 3 because the sum of its digits, 14, is not divisible by 3.

Although the pupils can not discover this test for themselves, they can easily verify its truth by dividing numbers such as 24, 63, 84, and 162 by 3 and also adding the digits of the numbers and dividing by three. In all these cases they find that both the number itself and the sum of its digits are evenly divisible by 3. But by dividing numbers such as 32, 83 and 145 by 3 and also dividing the sums of their digits by 3, they find that for all of these numbers neither the number itself or the sum of the digits is divisible by 3. From these and similar cases the truth of the method of testing stated above is easily seen.

Test for 4. Any number is divisible by 4 if the number formed by its last two digits is divisible by 4. Thus 1632 is divisible by 4 because 32, the number formed by its last two digits, is divisible by 4; but 7846 is not divisible by 4 because 46, the number formed by its last two digits is not divisible by 4.

The pupils can verify the truth of this test by taking numbers such as 724, 3428, and 3572; and others such as 342, 2574, and 7358. By dividing both the number itself and the number formed by the last two digits by 4, the truth of the test is easily verified.

Test for 5. Any number is divisible by 5 if it ends in a 5 or 0.

The pupils can easily discover this test by dividing various numbers such as 24, 32, 85, 71, 290, 393, and 735 by 5.

Test for 8. Any number is divisible by 8 if the number formed by its last three digits is divisible by 8. Thus 75744 is divisible by 8 because 744 is, but 375492 is not divisible by 8 because 492 is not.

The pupils can verify the truth of this test in a way similar to that used in verifying the test for 4.

Test for 9. Any number is divisible by 9 if the sum of its digits is divisible by 9. Thus 5787 is divisible by 9 because the sum of its digits, 27, is divisible by 9. But 3498 is not divisible by 9, because the sum of the digits, 24, is not divisible by 9.

The pupils can verify the truth of this test in the same way that they verify the truth of the test for 3.

Test for 10. Any number is divisible by 10 if it ends in a cipher.

The pupils can discover this test by dividing numbers such as 374, 870, 759, 3842, and 2700 by 10.

61. Proofs of the Tests. Although formal proofs of the tests of divisibility should not usually be given in the grades, the teacher should know why the tests are true. In the upper grades if the pupils see the need for an explanation and ask for it the teacher should be able to give one, even if the pupils do not understand it very well.

Proof of the Tests for 2, 5 and 10. Although usually stated in different forms the tests for 2, 5 and 10 are really identical, namely, "Any number is divisible by 2, 5, or 10 if its right hand digit is divisible by 2, 5, or 10." This is due to the fact that the base of our place value system of writing numbers is 10. Any number written in our system is composed of some number of 10's represented by all of the digits except the right hand digit, plus its right hand digit. Thus $3877 = 3870+7 = 387\times10+7$. It follows, therefore, that the part of any number represented by all of the digits but the last is always evenly divisible by 2, 5 and 10 because it is always a number of tens and 10 itself and therefore any number of tens, is evenly divisible by 2, 5 and 10. Whether or not the entire number is evenly divisible by 2, 5, and 10, will therefore depend upon whether the last digit is evenly divisible by these numbers.

Proof of the Test for 4. The part of any number written in our system that is represented by all but the last two digits is always a number of hundreds. Thus 385 in 38576, means 38500 or 385 hundreds. The number represented by all but the last two digits of any number is, therefore, divisible by 4, as it is always a number of hundreds and 100 and therefore any number of hundreds is evenly divisible by 4. Whether or not the entire number is evenly divisible by 4 depends, therefore, on the part of the number reprsented by the last two digits.

Proof of the Test 8. The proof of this test is similar to the two proofs just given. It is left to the student as an exercise.

Proof of the Tests for 9 and 3. To find out under what conditions a number is divisible by 9 we must separate the number into two parts, in such a way that the first part must always be divisible by 9 no matter what the digits may be. Then we know that the whole number will be divisible by 9 if and only if the other part is divisible by 9. Any number of four digits, such as 4756, can be written in the following forms.

$$
\begin{aligned}
4756 &= 4\times1000+7\times100+5\times10+6\times1 \\
&= 1000\times4+100\times7+10\times5+1\times6 \\
&= 1000\times\text{1st digit} + 100\times\text{2nd digit} + 10\times\text{3rd digit} + 1\times\text{4th digit} \\
&= (999\times\text{1st digit} + 99\times\text{2nd digit} + 9\times\text{3rd digit}) + (1\times\text{1st digit} + 1\times\text{2nd digit} + 1\times\text{3rd digit} + 1\times\text{4th digit})
\end{aligned}
$$

In the last form $1000\times$1st digit has been separated into $999\times$1st digit $+ 1\times$1st digit, and the $999\times$1st digit is placed in the first part of the number because it is divisible by 9, and $1\times$1st digit is placed in the second part because it is not necessarily divisible by 9, etc., for each of the other digits.

CHAPTER VI.

COMMON FRACTIONS

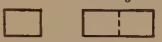
I. The Meaning of Fractions

62. The Fraction Ideas. The fraction idea, like the number idea, is a complex one and is a gradual growth as a result of the pupils' experience. The following outline shows approximately the stages in the development of the complete fraction idea.

I. Unit Fractions. ½, ⅓, ¼, etc.

a. Part of a single object, as

b. Part of a group of objects, as 0 0 0 0 0 0 0 0 0
(½ | ⅓)

c. Comparison of two single objects, as: Pint=½ of a quart. Foot=⅓ of a yard. One object ½ as large as another, as 

d. Comparison of two groups, 0 0 0 is ½ as many as 0 0 0 0 0 0.

All four of these ideas of the simple unit fractions should be started as early as possible, in the first grade if any number work is done there. The order in which the pupils obtain these four ideas of ½ is of no importance. Some teachers start with the comparison idea using the pint and the quart as the first introduction to the fraction idea. Most teachers prefer to start with the part idea, regarding it as more simple. When this is done the pupils' experience in sharing single objects and groups of objects with others can be made the introduction to fractions. The important thing is that the pupils in their concrete experiences with ½ meet all four of the ideas; ½ of an object, ½ of a group, one object ½ as large as another, and one group containing ½ as many objects as another group.

II. Simple Proper Fractions. Proper fractions that are not unit fractions, as: $\frac{2}{3}$, $\frac{2}{4}$, $\frac{3}{4}$.

a. Part of a single object, as

b. Part of a group 0 0 0 ′ 0 0 0 ′ 0 0 0
$\underbrace{\qquad\qquad}_{\frac{2}{3}}$

c. Comparison of two objects, as

The lower rectangle is $\frac{2}{3}$ of the upper.

d. Comparison of two groups. 000 000 000. Six is $\frac{2}{3}$ 000 000.

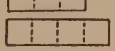
of nine. This work should start early in the third year.

III. Improper Fractions. $\frac{3}{2}$, $\frac{5}{4}$, etc. Improper fractions are first met late in the third or early in the fourth grade and arise from the work in addition of fractions, as $\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$. When encountered in this way their meaning should be made clear by picturing them objectively, using both single objects and groups as the unit.

a. Part of a single object.

b. Part of a group. $\overbrace{\underset{\smile}{00\ 00\ 00}}^{\qquad}$ $=\frac{4}{3}$ of 6.

Later the idea of one object being $\frac{4}{3}$ or $1\frac{1}{3}$ times as large as another object, and one group or number being $\frac{4}{3}$ or $1\frac{1}{3}$ times as large as another group or number must be introduced. If necessary, these ideas may also be objectified.

c. Comparison of two objects. 

The lower rectangle is $\frac{4}{3}$ of the upper.

d. Comparison of two groups. 00 00 00 Eight is 00 00 00 00

$\frac{4}{3}$ of six.

IV. Indicated Division. The idea of a fraction as an indicated division, as $12\frac{2}{3}=12\div 3$. This idea is first met in connec-

tion with reducing an improper fraction to a mixed number. The pupils already know that $1\frac{2}{3}=4$ or if they do not they can picture $1\frac{2}{3}$ as shown below.

They also know that $12\div 3=4$. Therefore it is evident that $1\frac{2}{3}=12\div 3$. From this and similar cases the pupils see that a fraction is an indicated division.

V. Concrete Number. The idea that a fraction is a concrete number, and that it can be dealt with as any other concrete number, is very useful in developing the work in fractions. Thus when the pupils first meet the addition of like fractions, such as $\frac{1}{5}+\frac{2}{5}$, they can reason it out because although they have never added fifths they have added other concrete numbers and $\frac{1}{5}+\frac{2}{5}$, or 1 fifth+2 fifths, is exactly the same kind of situation as 1 apple+2 apples with which situation the pupils are thoroughly familiar.

Exercise. Fraction Ideas

1. Represent the following fractions objectively using a single object as the unit.*

1. $\frac{7}{8}$ 2. $\frac{3}{10}$ 3. $\frac{5}{4}$ 4. $1\frac{1}{3}$

2. Represent the following fractions as indicated, using any convenient objects.

1. $\frac{5}{6}$ using a group of 12 objects as the unit.
2. $\frac{3}{5}$ using a group of 15 objects as the unit.
3. $\frac{5}{3}$ using a group of 9 objects as the unit.
4. $\frac{3}{4}$ using a group of 16 objects as the unit.

3. In each of the following tell how the first object or group of objects compares to the second. How the second compares to the first.

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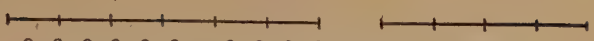
*At first pupils in the grades should use actual objects as apples, pieces of chalk to represent candy, etc. Later diagrams may be used as rectangles, lines or circles divided into parts.

1. 0 0 0 0; 0 0 0 0 0 0

2. 

3. 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0

4.

4. Draw a line 9 inches long. Under it draw a second line just $\frac{2}{3}$ as long.

5. Draw a rectangle of any convenient size. Draw a second rectangle just $\frac{5}{6}$ as large.

6. Take a group of 8 objects. Form another group containing $\frac{3}{4}$ as many objects.

7. Draw a line of any convenient length. Under it draw another line $\frac{5}{3}$ as long.

8. Take a group of 6 objects. Form another group containing $\frac{7}{3}$ as many objects.

63. Definitions. Three different definitions are commonly given for a fraction, based on the part idea, the comparison idea, and the indicated division idea, respectively.

A *fraction* is one or more of the equal parts into which some whole has been divided.

A *fraction* is a ratio or a comparison number which tells how one whole compares to another.

A *fraction* is an indicated division.

No one of these constitutes a complete definition of a fraction. Pupils in the grades must be familiar with all three of these different ideas. Most of the theory of fractions in the grades is, however, based on the part idea, that is on the first definition given. The terminology of fractions is also based on the part idea.

The word *fraction* comes from the Latin *frangere*, to break, and means literally a fragment or a portion. Fractions were at one time defined as broken numbers.

Denominator means, literally, that which gives a name. The denominator of a fraction, according to the part idea, tells the number of equal parts into which some whole has been divided and indirectly gives the name of the parts. Thus in $\frac{7}{8}$ the 8 gives the number of parts into which some whole is divided, namely 8, and since there are 8 parts each is called an eighth.

Numerator comes from the Latin word of the same form meaning one who numbers. The name comes from the fact that, according to the part idea of a fraction, the numerator tells the number of equal parts of some whole that have been taken. Thus in $\frac{7}{8}$ the numerator 7 gives the number of the equal parts that have been taken.

II. Principles

64. Statement and Proof of the Principles. An intelligent understanding of the work with fractions depends upon a knowledge of six fundamental principles. For convenience all six of these principles are given here. In the grades, of course, they should be met with and discovered one at a time, as needed. No proof or explanation should ordinarily be attempted when they are first presented.

I. Both the numerator and denominator of a fraction may be multiplied by the same number without changing the value of the fraction. The denominator of a fraction tells the number of equal parts into which some whole has been divided. If it is multiplied by 2 there will be twice as many equal parts in the given whole and the parts will consequently be just half as large. The numerator tells the number of parts taken and if it is multiplied by 2 there will be twice as many parts taken. Therefore, multiplying both the numerator and denominator by 2 does not change the value of the fraction as it gives twice as many parts taken but they are only half as large. Similar explanations could be given for multiplying both terms

by 3 or any other number. Explanations might also be based on the comparison and on the indicated division ideas.

II. Both terms of a fraction may be divided by the same number without changing the value of the fraction. The explanation of this and the following principles is left to the student as an exercise.

III. If the numerator of a fraction is multiplied by any number the value of the fraction is multiplied by that number.

IV. If the numerator of a fraction is divided by any number the value of the fraction is divided by that number.

V. If the denominator of a fraction is multiplied by any number the value of the fraction is divided by that number.

VI. If the denominator of a fraction is divided by any number the value of the fraction is multiplied by that number.

Exercise. Principles of Fractions

1. Explain why multiplying both the numerator and denominator of a fraction by 3 does not change the value.

2. State and explain the effect of dividing both terms of a fraction by 2. By 3.

3. State and explain the effect of multiplying the numerator of a fraction by 4.

4. State and explain the effect of dividing the numerator of a fraction by 2.

5. State and explain the effect of multiplying the denominator of a fraction by 3.

6. State and explain the effect of dividing the denominator of a fraction by 2.

7. Does adding the same number to both the numerator and denominator of a fraction change its value? Prove the truth of your answer by taking a special case.

8. Does subtracting the same number from both the numerator and denominator of a fraction change its value? Prove.

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II. Both terms of a fraction may be divided by the same number without changing the value of the fraction. The explanation of this and the following principles is left to the student as an exercise.

III. If the numerator of a fraction is multiplied by any number the value of the fraction is multiplied by that number.

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7. Does adding the same number to both the numerator and denominator of a fraction change its value? Prove the truth of your answer by taking a special case.

8. Does subtracting the same number from both the numerator and denominator of a fraction change its value? Prove.

III. History

65. Old Notations for Fractions. The subject of fractions was one of great difficulty for a long time. This was partly due to the lack of a convenient way of writing fractions. The Babylonians used only fractions with the denominator 60. In writing fractions the numerator was written and the denominator was understood. The Romans also usually kept the denominator constant but used 12 for the denominator instead of 60. They had no notation for fractions but gave each of the fractions $\frac{1}{12}$, $\frac{2}{12}$, $\frac{3}{12}$ . . . . . $\frac{11}{12}$, a different name and always wrote out the name in full. The Hindus wrote the numerator above the denominator as we do to-day, but did not use a dividing line. Thus $\frac{2}{3}$ was written $\frac{2}{3}$. In mixed numbers the integral part was written above, thus $1\frac{2}{3}$ was written $\frac{1}{2}$. The Arabs introduced the use of the line separating the numerator and denominator.

66. Methods of Calculating with Fractions. There has been but little change in the methods of calculating with fractions since the invention of the Hindu notation. Most of our methods to-day are substantially the same as those used by the Hindus, the Arabs, and the people of Europe during the middle ages. The following example shows the usual form for addition and subtraction in the sixteenth century. The common denominator was written below and makes $\frac{17}{12}$ the changed numerators above.

$$\begin{array}{cc} 8 & 9 \\ \hline \frac{2}{3} & \frac{3}{4} \\ \hline 12 & \end{array}$$

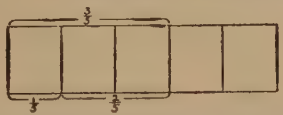
Multiplication in the middle ages was performed much the same as to-day. Three methods of division were in common use (1) By changing to common denominators and dividing the numerators as, $\frac{2}{3} \div \frac{1}{2} = \frac{4}{6} \div \frac{3}{6} = \frac{4}{3}$, (2) By inverting the divisor and multiplying, as $\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \times \frac{2}{1} = \frac{4}{3}$, and (3) By cross multiplication of the denominator of one and the numerator of the other. The divisor was written first and the numer-

ator and denominator of the result above and below, respectively. The work for $\frac{3}{4} \div \frac{2}{3}$ is given below. The divisor was written first. The $\times$ did not mean multiplication but simply indicated the cross products to be taken. The result is $\frac{9}{8}$.

$$\frac{2}{3} \times \frac{3}{4}$$
$$8 \quad\quad 9$$

IV. Addition and Subtraction of Fractions

67. Addition and Subtraction of Like Fractions. The work in addition and subtraction of fractions in the grades should start with simple cases such as $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$; $\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$; $\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$; etc. Such situations are met with objectively for some time before any formal work is done. Early in the fourth year the pupils should work out a general method of adding and subtracting fractions having the same denominators. They can take cases such as $\frac{1}{5}+\frac{2}{5}$ and $\frac{1}{6}+\frac{3}{6}$ and obtain the results by picturing the situation objectively, as



They can also obtain the results by thinking of the fractions in the same way as they think of concrete numbers. Thus

$\frac{1}{5}+\frac{2}{5}=?$
$1¢+2¢=3¢$
1 lb.+2 lbs.=3 lbs.
1 fifth+2 fifths=3 fifths.
$\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$

After working out several cases they should generalize in some such form as the following.

To add fractions having the same denominators, add the numerators and write the sum over the common denominator.

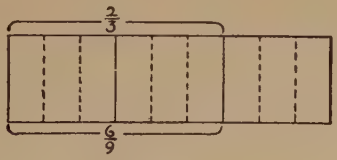
A similar generalization should also be made for the subtraction of like fractions.

68. Addition and Subtraction of Unlike Fractions. The addition and subtraction of unlike fractions involves two difficulties not present in the addition and subtraction of like fractions. These are (1) the determining of the common denominator, and (2) the changing of the fractions to equivalent fractions having the common denominator. Thus $\frac{3}{8}+\frac{5}{12}$ can not be added as they stand because they do not have the same name, that is, the same denominator. So we must determine some name (denominator) to which both fractions can be changed and then must change each fraction to an equivalent fraction having that name (denominator). The first cases the pupils meet are simple ones such as $\frac{1}{4}+\frac{1}{2}$, $\frac{1}{6}+\frac{1}{2}$, $\frac{1}{6}+\frac{2}{3}$, etc. From the objective work that has preceded this they know that halves can be changed to fourths and that $\frac{1}{2}=\frac{2}{4}$; that $\frac{1}{2}=\frac{3}{6}$, and that $\frac{2}{3}=\frac{4}{6}$.

Eventually, however, they will meet cases in which the common denominator and the equivalent fractions having that denominator are not known. In order to handle these they must discover some method of determining the common denominator and some method of changing the given fractions to equivalent fractions having that denominator.

69. Method of Changing Fractions to Equivalent Fractions Having a Different Denominator. In order to be able to change a fraction to an equivalent fraction having a different denominator the pupils must become acquainted with the principle that both the numerator and denominator of a fraction can be multiplied by the same number without changing the value. No general proof or explanation of the principle should be attempted but it should be discovered by means of simple cases. If the pupils multiply both terms of the fraction $\frac{1}{2}$ by 2, they get $\frac{2}{4}$ which they should already know is equal to $\frac{1}{2}$. If both terms of the fraction $\frac{2}{3}$ are multiplied by 3, the result is $\frac{6}{9}$.

Now, if the original fraction $\frac{2}{3}$ and the resulting fraction $\frac{6}{9}$ are pictured objectively, as shown below, they are seen to be equal.



By picturing out other similar cases the pupils soon discover the principle as stated above.

70. Methods of Finding the Least Common Denominator. As fractions are changed from the given denominator to the common denominator by *multiplying* both terms by the same number, it is evident that the common denominator is a *multiple* of all the given denominators and therefore it must be divisible by each of the given denominators. The least common denominator therefore will always be the smallest number that is divisible by each of the given denominators. Most of the fractions dealt with below the seventh grade are the simple fractions that are used in business and in practical affairs, namely the halves, thirds, fourths, fifths, sixths, eights, tenths, and twelfths. With these fractions the least common denominator should be found by inspection.

The easiest case is that in which the largest denominator is itself the least common denominator, as in the following.

$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{5}{12}$	$\frac{5}{12}$
$\frac{2}{9}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$
				$\frac{1}{3}$

In each of these the largest denominator is divisible by each of the other denominators, so it is the least common denominator.

Next the pupils should meet cases in which the least common denominator is the product of the denominators, as in the following.

$$\begin{array}{ccc} \frac{1}{3} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{5} & \frac{1}{5} \end{array}$$

In these the largest denominator is not divisible by the other denominator, nor by any factor of it, so the least common denominator is the product of the two denominators.

Finally cases such as the following should be considered.

$$\begin{array}{ccc} \frac{3}{8} & \frac{5}{12} & \frac{5}{6} \\ \frac{1}{6} & \frac{3}{8} & \frac{3}{4} \end{array}$$

In these the largest denominator does not contain the other denominator but does contain a factor of it. Thus in the first, the largest denominator 8, does not contain the other denominator 6, but does contain one factor of the 6, namely 2. In order to contain the 6 the least common denominator must have the other factor of the 6, namely 3, so it is evidently 8×3, or 24.

From the preceding cases the following method of procedure is derived.

To find the least common denominator start with the largest denominator. If it is divisible by each of the other denominators it is the least common denominator. If not, multiply the largest denominator by whatever factors are necessary in order that it may be divisible by each of the other denominators.

Much practice must be given on finding the least common denominator by inspection.

In the seventh and eighth grades some work is usually done with fractions having larger denominators and a written method of finding the least common denominator must be used. The best written method is given below. It is simply the inspection method described above written out in detail.

Example 1.

$17/84+31/63+107/126$

Work of Finding L. C. D.

$84=2\times2\times3\times7$
$63=3\times3\times7$
$126=2\times3\times3\times7$
L. C. D. $=2\times3\times3\times7\times2$
$=126\times2=252$

Explanation. To contain 126, the largest denominator, the L. C. D. must have all the factors of 126. So start the L. C. D. by writing down the factors of 126, namely $2\times3\times3\times7$. To contain 84 there must be another factor 2, so this is added to the L. C. D. No other factors are necessary in order to contain the 63.

Example 2.

$109/125+9/35+113/225$

Work of Finding L. C. D.

$125=5\times5\times5$
$35=5\times7$
$225=5\times5\times3\times3$
L. C. D. $=5\times5\times3\times3\times5\times7$
$=225\times35=7875$

71. Form of Work in Addition and Subtraction. Two forms are used in addition and subtraction of fractions, namely the horizontal and the vertical. Of these the vertical form is used the most in arithmetic.

The horizontal form is as follows:

$$3/8+5/6=9/24+20/24=29/24=15/24$$

The full vertical form of the same example is:

$3/8=9/24$
$5/6=20/24$
$29/24=15/24$

As soon as the steps involved are thoroughly understood the vertical form should be abbreviated as follows:

L. C. D. 24	
$3/8$	9
$5/6$	20
$15/24$	$29/24$

Exercise. Addition and Subtraction of Fractions

1. Find the L. C. D. in each of the following by inspection.

1.	2.	3.	4.	5.	6.	7.
$\frac{3}{8}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{11}{21}$	$\frac{7}{10}$	$\frac{13}{15}$
$\frac{1}{2}$	$\frac{3}{14}$	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{3}{14}$	$\frac{8}{15}$	$\frac{3}{4}$
$\frac{7}{16}$	$\frac{11}{21}$	$\frac{7}{24}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{5}{9}$	$\frac{2}{5}$
$\frac{3}{4}$	42	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	90	$\frac{5}{12}$

8.	9.	10.	11.	12.
$\frac{3}{4}$	$\frac{5}{6}$	$\frac{1}{5}$	$\frac{5}{7}$	$\frac{1}{9}$
$\frac{11}{18}$	$\frac{3}{8}$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{7}{8}$
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{11}{12}$
	$\frac{15}{16}$	30	63	72

2. Use the horizontal form in the following. Find the L. C. D. by inspection.

1. $\frac{1}{3}+\frac{2}{5}+\frac{1}{2}$
2. $\frac{5}{6}+\frac{3}{4}+\frac{7}{8}$
3. $\frac{7}{12}-\frac{4}{15}$

3. Use the full vertical form in the following. Find the L. C. D. by inspection.

1. $\frac{3}{8}+\frac{5}{12}$ 2. $\frac{11}{15}-\frac{7}{16}$

4. Rewrite the solutions to Exercise 3 in the abbreviated vertical form.

5. Use the abbreviated vertical form for the following. Find the L. C. D. by the written method described above.

1. $\frac{117}{132}+\frac{75}{88}+\frac{5}{24}$

Solution:

L. C. D.=264

$\frac{117}{132}$	234
$\frac{75}{88}$	225
$\frac{5}{24}$	55
$1\frac{125}{132}$	$\frac{514}{264}=1\frac{250}{264}$

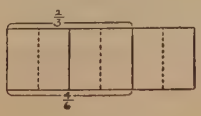
$132=2\times2\times3\times11$
$88=2\times2\times2\times11$
$24=2\times2\times2\times3$
L. C. D.$=2\times2\times3\times11\times2$
$=132\times2=264$

Explanation: The first denominator 132, contains all the factors of the L. C. D. except 2, so the numerator and denominator of the first fraction must be multiplied by 2. The second denominator contains all the factors of the L. C. D. but 3, so the terms of the second fraction are multiplied by 3. The last denominator contains all the factors of the L. C. D. but 11, so the terms of the last fraction are multiplied by 11.

2. $^{15}\!/_{72}+{}^{81}\!/_{96}+{}^{45}\!/_{184}$
3. $^{5}\!/_{18}+{}^{23}\!/_{24}+{}^{53}\!/_{84}$
4. $^{49}\!/_{52}-{}^{31}\!/_{91}$
5. $^{57}\!/_{66}-{}^{17}\!/_{78}$
6. $^{15}\!/_{84}+{}^{107}\!/_{140}+{}^{23}\!/_{60}$
7. $^{115}\!/_{126}+{}^{45}\!/_{81}+{}^{31}\!/_{189}$
8. $^{37}\!/_{132}+{}^{83}\!/_{154}+{}^{107}\!/_{231}$
9. $^{77}\!/_{102}+{}^{83}\!/_{136}+{}^{65}\!/_{68}$
10. $^{19}\!/_{33}-{}^{17}\!/_{77}$
11. $^{16}\!/_{81}-{}^{11}\!/_{126}$
12. $^{113}\!/_{120}+{}^{11}\!/_{72}+{}^{157}\!/_{180}$

V. Reduction of Fractions to Lowest Terms

72. Reduction to Lowest Terms. Very early in the work of adding and subtracting fractions the pupils will get results such as $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$; $\frac{1}{8}+\frac{3}{8}=\frac{4}{8}$; $\frac{5}{6}-\frac{2}{6}=\frac{3}{6}$. As soon as such fractions are obtained the pupils should learn to reduce them to lowest terms, that is, to equivalent fractions having the smallest possible numerator and denominator. At first the pupils will use the equivalents that have been worked out objectively and memorized, such as $\frac{4}{8}=\frac{1}{2}$. Eventually, however, in order to be able to reduce all cases they must become acquainted with the principle that dividing both the numerator and denominator of a fraction by the same number does not change the value of the fraction. This principle should be worked out objectively. If the pupils start with the fraction $\frac{4}{6}$ and divide both the numerator and denominator by 2, the result is $\frac{2}{3}$. If the $\frac{4}{6}$ and $\frac{2}{3}$ are represented objectively, as below, they are seen to be equal.



The general principle is derived by picturing several cases such as this. The teacher must insist that all fractional results be left in lowest terms until the habit is thoroughly established. In the written work of reducing fractions, the number by which the numerator and denominator are divided should not be written. It takes too long to write $\frac{15}{18} = \frac{15 \div 3}{18 \div 3} = \frac{5}{6}$. It is sufficient to write $\frac{15}{18} = \frac{5}{6}$. The notation $3\left|\frac{15}{18} = \frac{5}{6}\right.$, which is frequently encountered in some parts of the country, is not an accepted notation and should not be used.

VI. Multiplication and Division of Fractions

73. Order of Teaching. The work in multiplication and division of fractions should be started before addition and subtraction are completed. The multiplication and division of a fraction by a simple integer is just as easy as the simplest work in addition and subtraction and is needed just as early, and consequently should be taught before addition and subtraction of unlike fractions. The order of presenting the work in multiplication and division should be as follows. (1) Multiplication and division of fractions by simple integers, (2) Multiplication by a fraction, and (3) Division by a fraction.

74. Multiplication of Fractions by Integers. By working out cases such as $2 \times \frac{1}{3}$, $3 \times \frac{2}{3}$ and $4 \times \frac{2}{9}$ the pupils should discover a general method of multiplying a fraction by an integer. These cases may be worked out in several ways. One way is to make use of the meaning of multiplication.

$$2 \times \tfrac{1}{3} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$
$$3 \times \tfrac{2}{3} = \tfrac{2}{3} + \tfrac{2}{3} + \tfrac{2}{3} = \tfrac{6}{3}$$

A second, and very useful way, is to think of the fractions as concrete numbers.

$2 \times \frac{1}{3} = ?$
$2 \times 1¢ = 2¢$
2×1 lb. $= 2$ lbs.
2×1 third $= 2$ thirds.
$2 \times \frac{1}{3} = \frac{2}{3}$

After the results of several simple cases have been obtained by either method suggested above, the pupils should form a rule as follows.

To multiply a fraction by an integer, multiply the numerator by the integer.

Later it might be worth while to teach a second method of multiplying a fraction by an integer.

$$2 \times \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$$
$$3 \times \frac{5}{18} = \frac{15}{18} = \frac{5}{6}$$

In each of these cases, instead of first multiplying the numerator by the integer and then reducing the result, it would have been shorter to divide the denominator by the integer. From such cases the pupils can derive a second method of multiplying a fraction by an integer, which should be combined with the first method as follows.

To multiply a fraction by an integer, divide the denominator, or multiply the numerator by the integer.

The second method is not absolutely necessary as a working method but it involves one of the fundamental principles of fractions, with all of which the pupils should become familiar.

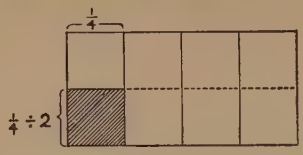
75. Division of Fractions by Integers.

$\frac{4}{5} \div 2 = ?$
$4¢ \div 2 = 2¢$
4 fifths $\div 2 = 2$ fifths
$\frac{4}{5} \div 2 = \frac{2}{5}$

From cases such as this the pupils form the generalization.

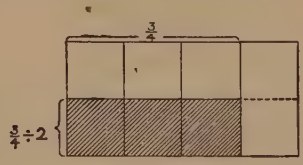
To divide a fraction by an integer, divide the numerator by the integer.

As the numerator can not always be divided by the integer, this method alone is not sufficient. The pupils must find out how to handle cases such as $\frac{1}{4} \div 2$ and $\frac{3}{4} \div 2$. These might be pictured out as follows. To divide $\frac{1}{4}$ by 2 means to divide it into two equal parts or to find half of it.



From the diagram it is seen that $\frac{1}{4} \div 2 = \frac{1}{8}$

To divide $\frac{3}{4}$ by 2 means to find half of it.



From the diagram, it is seen that $\frac{3}{4} \div 2 = \frac{3}{8}$.

From these and similar cases the pupils find another method of dividing a fraction by an integer, which combined with the first gives the following.

To divide a fraction by an integer, divide the numerator or multiply the denominator by the integer.

The division of a fraction by an integer might be postponed until after the pupils learn how to divide by a fraction and be made a special case under division by a fraction. This is probably not the best method for several reasons. Division of fractions by integers is easier than division by a fraction and the need for it arises sooner. Further, it is too long to have to think $\frac{3}{4} \div 2 = \frac{3}{4} \div \frac{2}{1} = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$. Finally division of fractions by integers involves two of the fundamental principles of fractions with all of which the pupils should be familiar.

76. Meaning of Multiplication by a Fraction. Multiplication by a fraction does not mean the same thing as multiplication by a whole number. 3×12 means three 12's, or $12+12+12$. A similar meaning cannot be given to $\frac{2}{3}\times12$ as we cannot have two-third 12's. The elementary meaning of multiplication must, therefore, be extended in order to include multiplication by a fraction. Any meaning may be given to multiplication by a fraction that will make this new kind of multiplication obey the same laws as multiplication of integers. It is a fundamental law of multiplication of integers that, in the case of abstract numbers, the multiplier and multiplicand can be interchanged without changing the result. Thus $3\times4=4\times3$. Multiplication by a fraction must obey this same law, so, whatever meaning is given to $\frac{2}{3}\times12$ it must be such as to make $\frac{2}{3}\times12=12\times\frac{2}{3}$. But $12\times\frac{2}{3}=2\frac{2}{3}=8$, so $\frac{2}{3}\times12$ must also be equal to 8. But 8 is $\frac{2}{3}$ of 12, so the following is a convenient *definition* to adopt for multiplication by a fraction.

To multiply by a fraction means to take that fractional part of the multiplicand.

The discussion of the meaning of multiplication by a fraction, given above, is not intended for grade pupils. As early as the first or second grades the pupils start finding fractional parts of numbers objectively. In the third grade they learn to find $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ etc. of numbers by dividing by 2, 3, or 4. In the fourth grade they learn to find $\frac{2}{3}$, etc., of numbers by dividing by 3 and multiplying by 2. After pupils have had some practice in finding parts of numbers they should be told that $\frac{2}{3}$ of 9 may also be written $\frac{2}{3}\times9$ and from that time on should use "$\times$" and the "of" interchangeably. When they find how to multiply a fraction by an integer, as $3\times\frac{2}{7}=\frac{6}{7}$ and $2\times\frac{3}{8}=\frac{3}{4}$, they should apply the same methods to multiplication of an integer by a fraction, as $\frac{2}{7}\times3=\frac{6}{7}$ and $\frac{3}{8}\times2=\frac{3}{4}$.

77. Multiplication of a Fraction by a Fraction. The pupils can discover the rule for multiplying a fraction by a fraction in several ways, two of which are given below.

$\frac{1}{3}\times\frac{2}{3}=\frac{1}{3}$ of $\frac{2}{3}=\frac{2}{3}\div 3=\frac{2}{9}$

$\therefore\ \frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$

$\frac{2}{3}\times\frac{2}{5}=\frac{2}{3}$ of $\frac{2}{5}$

$\frac{1}{3}$ of $\frac{2}{5}=\frac{2}{5}\div 3=\frac{2}{15}$

$\frac{2}{3}$ of $\frac{2}{5}=2\times\frac{2}{15}=\frac{4}{15}$

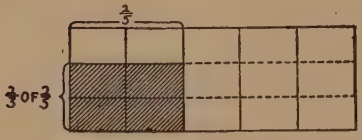
$\therefore\ \frac{2}{3}\times\frac{2}{5}=\frac{4}{15}$

The results in the two cases given above might also be obtained by means of diagrams.

$\frac{1}{3}\times\frac{2}{3}=\frac{1}{3}$ of $\frac{2}{3}$

From the diagram $\frac{1}{3}\times\frac{2}{3}$ or $\frac{1}{3}$ of $\frac{2}{3}$ is seen to be $\frac{2}{9}$ of the whole.

$\frac{2}{3}\times\frac{2}{5}=\frac{2}{3}$ of $\frac{2}{5}$



From the diagram $\frac{2}{3}\times\frac{2}{5}$ or $\frac{2}{3}$ of $\frac{2}{5}$ is seen to be $\frac{4}{15}$ of the whole.

From the two cases $\frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$, $\frac{2}{3}\times\frac{2}{5}=\frac{4}{9}$, and other similar cases the pupils can discover the following short method of multiplying a fraction by a fraction.

To multiply a fraction by a fraction, multiply the numerators for the numerator of the product and multiply the denominators for the denominator.

78. Division by a Fraction. Many methods of dividing by a fraction have been in use at various times.* Of these the

*See Section 66 for some of the older methods.

one that has proved most satisfactory is the method of inverting the divisor. In many schools pupils are first taught to divide fractions by changing to a common denominator and dividing the numerators, and later taught to invert. Where this is done, the original habit often persists and interfers with the new. To avoid this, the pupils should be taught to invert and multiply from the very start.

The following should make the reason for inverting and multiplying clear to the teacher. In the example $12 \div \frac{2}{3}$, the question is "How many times is $\frac{2}{3}$ contained in 12?" In reasoning this out, first think "$\frac{1}{3}$ is contained 3 times in 1, and 12×3 times or 36 times in 12. But $\frac{2}{3}$ is twice as large as $\frac{1}{3}$ and so will be contained in 12 just half as many times as $\frac{1}{3}$, or $36 \div 2 = 18$ times." It is seen from the above that in dividing by $\frac{2}{3}$, we must do two things (1) multiply by 3 (the denominator) in order to find out how many times $\frac{1}{3}$ is contained in the dividend, and (2) divide by 2 (the numerator) in order to find how many times $\frac{2}{3}$ is contained. That is, in dividing by a fraction, we must divide by its numerator and multiply by its denominator. The rule for dividing by a fraction was formerly often given in this form. In practice, however, the easiest way to divide by the numerator and multiply by the denominator is to invert and multiply. Thus in $12 \div \frac{2}{3}$ we must divide by the numerator 2 and multiply by the denominator 3, which is just what we do if we invert and multiply as $12 \times \frac{3}{2}$.

Probably the best way to develop the method of inversion in the grades is to base it upon the fact that multiplication and division are inverse processes, dividing by any number gives the same result as multiplying by that number inverted. The pupils already know that $12 \div 2$ means the same thing as $\frac{1}{2}$ of 12 or $\frac{1}{2} \times 12$, that $6 \div 3 = \frac{1}{3} \times 6$, etc. This should be reviewed at this time. Next, the pupils should reason out such cases as $4 \div \frac{1}{2}$ and $5 \div \frac{1}{3}$. The example $4 \div \frac{1}{2}$ asks "how

many ½'s are there in 4?" The pupils can reason this out as they know that there are two ½'s in 1, so in 4 there are 4×2 or 8 halves. So $4\div\frac{1}{2}=8$. But $8=2\times4$, so $4\div\frac{1}{2}=2\times4$. In the same way the pupils can reason out that $5\div\frac{1}{3}=15$ and therefore $5\div\frac{1}{3}=3\times5$.

The pupils now have the following information which should be summarized on the board.

$12\div2=\frac{1}{2}\times12$	$4\div\frac{1}{2}=2\times4$
$6\div3=\frac{1}{3}\times6$	$5\div\frac{1}{3}=3\times5$
Etc.	Etc.

If ½ and ⅓ are turned upside down or inverted they become $\frac{2}{1}$ and $\frac{3}{1}$ or 2 and 3, respectively. If 2 and 3, or $\frac{2}{1}$ and $\frac{3}{1}$ are inverted they become ½ and ⅓. So the pupils can form the generalization "to divide by any number is the same as to multiply by that number inverted." And from this they can form the following working rule for dividing by a fraction.

To divide by a fraction invert and multiply.

79. Cancellation. Very early in the multiplication and division of fractions, the pupils should be taught to shorten the work by means of cancellation. Cancellation should be taught as an application of the principle that both the numerator and denominator of a fraction may be divided by the same number without changing the value.

(1) $8\times\frac{5}{12}=\frac{40}{12}=1\frac{4}{12}$

(2) $\frac{3}{4}\times\frac{8}{9}=\frac{24}{36}=\frac{6}{9}=\frac{2}{3}$

In (1) the work may be shortened by dividing out the common factor 4 from the numerator and denominator before multiplying instead of after.

$$\overset{2}{\cancel{8}}\times\frac{5}{\cancel{12}_{3}}=\frac{10}{3}$$

The pupils should think $12 \div 4 = 3$ (cancelling the 12 and writing the 3), $8 \div 4 = 2$ (cancelling the 8 and writing the 2).

In (2) the work may be shortened by dividing out the common factors 4 and 3 before multiplying.

$$\frac{\cancel{3}^{1}}{\cancel{4}_{1}} \times \frac{\cancel{8}^{2}}{\cancel{9}_{3}} = \frac{2}{3}$$

For awhile at least, the 1's should be written in order to emphasize that cancelling is dividing common factors out of the numerator and denominator.

Too often pupils get the idea that cancelling is "crossing out equal numbers." This leads to many mistakes both in arithmetic and algebra. The following are the commonest mistakes.

1. $$\frac{\cancel{2}}{\cancel{3}} \times \frac{\overset{\cancel{3}}{\cancel{6}}}{\cancel{5}} \times \frac{\cancel{5}}{\underset{\cancel{2}}{4}} = 0$$

As everything is "crossed out" and nothing left the pupils conclude the result is 0.

2. $$\frac{x+\cancel{2}}{y+\cancel{2}} = \frac{x}{y}$$

This is one of the commonest mistakes in algebra. The students see a 2 in both the numerator and denominator and cross it out. This mistake is due to the fact that they have not been taught the true meaning of cancellation.

EXERCISE. DEVELOPING THE PROCESSES WITH FRACTIONS.

1. Develop each of the following in detail as you would in the grades. Use different examples from those given in the text and more of them.

1. The rule for adding like fractions.
2. The principle used in changing a fraction to an equivalent fraction having a different denominator.
3. The principle used in changing fractions to lower terms.
4. The two methods of multiplying a fraction by an integer.
5. The two methods of dividing a fraction by an integer.
6. The rule for multiplying a fraction by a fraction.
7. The rule for dividing by a fraction.

VII. Mixed Numbers and Improper Fractions

80. Order of Teaching. The work with mixed numbers in the grades should be taken up along with the corresponding work with pure fractions. It is separated here in order to simplify the discussion.

81. Changing Improper Fractions to Mixed Numbers. In adding fractions, pupils soon get results in which the numerator is larger than the denominator, as $\frac{7}{8}+\frac{1}{2}=\frac{7}{8}+\frac{4}{8}=1\frac{1}{8}$. As soon as such cases are met with, the name improper fraction should be given and the meaning made clear by objective representation.

From the diagram, it is seen that $1\frac{1}{8}$ is one whole and $\frac{1}{8}$ of another whole, that is $1\frac{1}{8}=1\frac{1}{8}$.

The pupils next learn how to change an improper fraction to a mixed number. It is in this connection that they first learn that a fraction is an indicated division. By drawing a diagram, if necessary, the pupils find that $1\frac{2}{2}=4$ and $1\frac{7}{5}=3\frac{2}{5}$; but $12\div 3=4$ and $17\div 5=3\frac{2}{5}$. From such cases it is seen that a fraction is an indicated division and, that,

To change an improper fraction to a whole or a mixed number, divide the numerator by the denominator.

82. Changing Mixed Numbers to Improper Fractions. A mixed number is expressed as so many wholes and some fractional part of a whole. Thus $2\frac{3}{4}$ means 2 wholes and $\frac{3}{4}$ of another whole. To change $2\frac{3}{4}$ to an improper fraction, we must express the whole thing as some number of fourths. One whole equals $\frac{4}{4}$, so 2 wholes equals $\frac{8}{4}$. So $2\frac{3}{4}=\frac{8}{4}+\frac{3}{4}=\frac{11}{4}$. By reasoning out several such cases, the pupils form the following generalization.

To change a mixed number to an improper fraction, multiply the integer by the denominator of the fraction, add the numerator and write the result over the denominator.

83. Addition of Mixed Numbers. The addition of mixed numbers presents no difficulties. The vertical form should be used for the work. At first the full form should be used but as soon as possible the work should be abbreviated.

Full Form.

$$\begin{array}{r} 12\frac{2}{3}=12\frac{8}{12} \\ 9\frac{3}{4}=\ 9\frac{9}{12} \\ \hline 21\,\frac{17}{12}=22\frac{5}{12} \end{array}$$

Abbreviated Form.

L. C. D.=12

$12\frac{2}{3}$	8
$9\frac{3}{4}$	9
$22\frac{5}{12}$	$\frac{17}{12}=1\frac{5}{12}$

84. Subtraction of Mixed Numbers. The subtraction of mixed numbers presents one difficulty, namely the difficulty of having a larger fraction in the subtrahend than in the minuend. This is the same difficulty that occurs in the subtraction of whole numbers and may be overcome in the same two ways, namely either by borrowing or by equal-additions.* The same method should be used in subtracting mixed numbers as in the subtraction of integers. Whichever method is used, the full form should be used until the steps are thoroughly understood, when the abbreviated form should be substituted.

*See Section 41.

Full Form Using Borrowing.

$$\begin{array}{r} 37\frac{1}{3}=37\frac{4}{12}=36\frac{16}{12} \\ 12\frac{3}{4}=12\frac{9}{12}=12\frac{9}{12} \\ \hline 24\frac{7}{12} \end{array}$$

As soon as possible, the last step should be omitted in the written solution and the borrowing performed mentally as with integers. The work can then be put in the following abbreviated form.

Abbreviated Form.

L. C. D.=12	
$37\frac{1}{3}$	4
$12\frac{3}{4}$	9
$24\frac{7}{12}$	

Full Form Using Equal-Additions.

$37\frac{1}{3}=37\frac{4}{12}$	$37\frac{16}{12}$
$12\frac{3}{4}=12\frac{9}{12}$	$13\frac{9}{12}$
	$24\frac{7}{12}$

As soon as possible the third step should be omitted from the written solution and the adding of the $1\frac{2}{12}$ to the minuend and the 1 to the subtrahend performed mentally. When this is done, the abbreviated form given above can be used.

It should be noticed that the final abbreviated form is the same whether borrowing or equal-additions is used. The mental steps are, however, different. In borrowing, the pupil mentally changes the 4 (twelfths) to 16 (twelfths) and 37 to 36. In using the equal-additions plan, the pupil changes the 4 (twelfths) to 16 (twelfths) and the 12 to 13.

85. Multiplication of Mixed Numbers. Three cases are usually considered in the multiplication of mixed numbers (1) Multiplying a mixed number by an integer, (2) Multiplying an integer by a mixed number, and (3) Multiplying a mixed number by a mixed number.

The following form should be used in multiplying a mixed number by an integer.

$$\begin{array}{r} 23\frac{1}{2} \\ 27 \\ \hline 13\frac{1}{2} \\ 161 \\ 46 \\ \hline 634\frac{1}{2} \end{array} \qquad 27\times\tfrac{1}{2}=\tfrac{27}{2}=13\tfrac{1}{2}$$

Sometimes it is easier to multiply the integer by the mixed number.

$$\begin{array}{r} 236 \\ 38\frac{2}{3} \\ \hline 3\overline{)472} \\ \hline 157\frac{1}{3} \\ 1888 \\ 708 \\ \hline 9125\frac{1}{3} \end{array} \quad \text{or} \quad \begin{array}{r} 236 \\ 38\frac{2}{3} \\ \hline 1888 \\ 708 \\ 157\frac{1}{3} \\ \hline 9125\frac{1}{3} \end{array} \qquad \tfrac{2}{3}\times 236=\tfrac{472}{3}=157\tfrac{1}{3}$$

In multiplying a mixed number by a mixed number, the mixed numbers should be changed to improper fractions.

$$8\tfrac{2}{3}\times 7\tfrac{1}{2}=\frac{\cancel{26}^{13}}{\cancel{3}}\times\frac{\cancel{15}^{5}}{\cancel{2}}=65$$

$$13\tfrac{1}{4}\times 8\tfrac{1}{3}=\tfrac{53}{4}\times\tfrac{25}{3}=\tfrac{1325}{12}=110\tfrac{5}{12}$$

If desired, all three of these cases may be handled in the same way, namely by changing the mixed numbers to improper fractions.

86. Division of Mixed Numbers. In any division involving mixed numbers, they should always first be reduced to improper fractions.

$$18\tfrac{3}{4}\div 7\tfrac{2}{3}=\tfrac{75}{4}\div\tfrac{23}{3}=\tfrac{75}{4}\times\tfrac{3}{23}=\tfrac{225}{92}=2\tfrac{41}{92}$$

Exercise. Mixed Numbers and Improper Fractions

1. Develop the following as you would in the grades. Use different examples from those given in the text and more of them.

1. The method of changing an improper fraction to a mixed number.
2. The method of changing a mixed number to an improper fraction.

2. Subtract, using the borrowing plan. Write out the full form of solution.

1. $287\frac{1}{2}$ $\quad$ 2. $359\frac{2}{3}$
$\underline{132\frac{3}{5}}$ $\quad$ $\underline{113\frac{7}{8}}$

3. Subtract the examples in Exercise 2, using the equal-additions plan. Write out the complete form of solution.

4. Add and subtract the following, as indicated, using the abbreviated form.

1. $386\frac{3}{4}+759\frac{2}{3}$ $\quad$ 3. $493\frac{5}{12}+89\frac{7}{15}$
2. $584\frac{3}{5}-291\frac{7}{8}$ $\quad$ 4. $917\frac{5}{6}-374\frac{7}{9}$

5. Multiply:

1. $239\frac{2}{3}$ $\quad$ 3. 529
$\underline{37}$ $\quad$ $\underline{18\frac{2}{3}}$
2. $817\frac{3}{4}$ $\quad$ 4. 3756
$\underline{83}$ $\quad$ $\underline{7\frac{4}{5}}$

5. $27\frac{1}{2}\times32\frac{3}{4}$
6. $87\frac{1}{3}\times52\frac{1}{4}$

6. Divide:

1. $87\div9\frac{3}{4}$ $\quad$ 3. $46\frac{1}{2}\div15\frac{2}{3}$
2. $32\frac{1}{2}\div7$ $\quad$ 4. $7\frac{3}{8}\div9\frac{1}{5}$

CHAPTER VII.

DECIMAL FRACTIONS

I. The Reading and Writing of Decimals

87. The Decimal Notation. By a simple extension of the Hindu system of writing whole numbers we obtain an easy way of writing certain fractions and, as a result, an easy way of calculating with these fractions. The Hindu system is a place value system having ten as a base. Any digit placed in the right hand place represents that number of units, in the next place to the left that number of tens, etc., each place to the left having a value ten times that of the preceding place. In this way the system can be extended indefinitely to the left.

For a long time the world stopped with units place at the right. Finally, however, it was seen that if the system were extended to the right of units place it would give an easy way of writing certain fractions. Starting at any place in the system and going to the *right* each place has a value just one-tenth of that of the preceding place. So if the system is extended to the right of units place according to the same laws, the next place will have a value just one-tenth as great as that of units place or a value equal to $\frac{1}{10}$ of 1 or $\frac{1}{10}$. The next place to the right will have a value of $\frac{1}{10}$ of $\frac{1}{10}$ or $\frac{1}{100}$, etc. In this way the system can be extended indefinitely to the right.

Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units	Tenths	Hundredths	Thousandths	Ten Thousandths	Hundred Thousandths
100,000	10,000	1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10,000}$	$\frac{1}{100,000}$
X	X	X	X	X	X	X	X	X	X	X

As long as the right hand place was always units place there was always a fixed point of reference from which the value of each of the places could be determined. When the system was extended to the right, units place was no longer fixed and it became necessary to put in some point of reference. The most convenient place to put this reference point was between the integral and the fractional places, that is, immediately after units place. In America a period placed on a line with the bottom of the digits is used for this purpose. It is called a *decimal point.* Thus 3.7 means $3\frac{7}{10}$. In England the point is put on a line with the centers of the digits as 3·7. In Germany and France the comma is used instead of of the period, as 3,7.

By extending the system to the right of units place, according to the same laws as govern the rest of the system, places are obtained whose values are fractions having the various powers of ten for denominators. As a result this notation affords an easy way to write such fractions. Thus $\frac{7}{10}$ can be written by placing 7 in the place whose value is $\frac{1}{10}$, that is in the first place to the right of the decimal point, as .7; and $\frac{3}{100}$ can be written by placing 3 in the place whose value is $\frac{1}{100}$, that is in the second place to the right of the decimal point, as .03, etc.

It is especially important to notice that the system is uniform from end to end and obeys the same laws in one part as in another. There is no break in the system at the decimal point or any other place.

88. Definition. A *decimal fraction,* or a *decimal,* is a fraction whose denominator is some power of ten, and which is written by writing the numerator in the proper place in our decimal place value notation.

89. The Reading of Decimals. A pure decimal such as .346 is read by first reading the number as if it were a whole number (this is the numerator of the fraction), and then

reading the denominator, which is obtained from the number of decimal places. Thus .346 is read "three hundred forty-six thousandths." In order to read decimals readily the denominators corresponding to a given number of decimal places must be memorized. Pupils in the grades should memorize that one decimal place means tenths, two means hundredths, three means thousandths, and four means ten thousandths. The places beyond this need not be memorized but the pupils should be able to determine the names if necessary.

In reading a mixed number written in the decimal notation, as 372.745, the whole number should be read first followed by the decimal read in the usual way, the two being separated by the word "and." Thus 372.745 is read "three hundred seventy-two *and* seven hundred forty-five thousandths. Care should be taken to use the "and" only between the integer and fraction as it serves the same purpose in reading decimals that the decimal point does in writing them. If "and" is used between the parts of the whole number confusion often results. Thus if .374 is improperly read "three hundred and seventy-four thousands" it is apt to be confused with 300.074 which is properly read "three hundred and seventy-four thousandths." If .374 is read correctly, namely "three hundred seventy-four thousandths," no confusion can result.

Pupils should also be practiced on reading mixed decimals as improper fractions. Thus 3.2 is 32 tenths, 37.5 is 375 tenths, 8.41 is 841 hundredths and 51.67 is 5,176 hundredths.

Although pupils should be able to read decimals in the ways described above they should also be familiar with the method commonly used in practical affairs — reading the digits singly or in pairs in order and saying "point" at the decimal point. Thus 3894.76 can be read "thirty-eight, ninety-four, point, seventy-six," or "three, eight, nine, four, *point*, seven, six."

If a common fraction is written in connection with a decimal, as in .2½, it does not occupy a decimal place by itself but always goes with the preceding place. Thus .2½ does not mean 2 tenths and ½ of a hundredth but 2½ tenths, that is both the 2 and the ½ are considered as being in tenths place.

.2½ is read "2½ tenths"
.0½ is read "½ of a tenth"
2.0½ is read "2 *and* ½ of a tenth"

EXERCISE. READING AND WRITING DECIMALS

1. Read the following decimals.

1. .3785	6. .3895	11. .00½
2. .751	7. 3000.0895	12. .0000⅓
3. .7095768	8. .275	13. .07½
4. 587.8976	9. 200.075	14. 7.00½
5. 3757.892	10. .02½	

2. Read the following as mixed numbers and as improper fractions.

1. 3.2	3. 45.3	5. 3.427
2. 2.56	4. 75.68	6. 27.128

3. Write the following.

1. Seven thousand *and* three hundred fifty-nine ten thousandths.
2. Seven thousand three hundred fifty-nine ten thousandths.
3. Forty-seven tenths.
4. Seven hundred forty-eight hundredths.
5. Seven hundred *and* forty-eight hundredths.
6. Two thousand eight hundred thirteen hundredths.
7. Two and one-fifth tenths.
8. Two *and* one-fifth of a tenth .
9. One-third of a hundredth.
10. Three *and* three and one-half tenths.

II. History of Decimals

90. The Invention of Decimals. Decimal fractions are so simple and convenient and such a natural outgrowth of the Hindu notation that it is remarkable that they were not discovered earlier. Although many mathematicians came very close to making the discovery at various times, the actual discovery or invention is probably due to Simon Stevin of Bruges in Belgium, who in 1585 published a work of only seven pages, in which decimal fractions were explained.

In spite of the great simplicity of decimal fractions, the world has been very slow to adopt them. The invention of the metric system of weights and measures was a big step in this direction but even to-day these are not in common use in America and England.

91. Methods of Writing Decimals. As has already been mentioned, several methods of writing decimals are in common use to-day.* In the past many different methods have been used. Stevin placed a cipher above units place, a 1 above tenths, a 2 above hundredths, etc. Thus 32.576 would have been written by Stevin $32^{0\,1\,2\,3}576$. Other early notations were $3^{0}\,.\,7^{1}\,.\,5^{11}\,.\,4^{111}$ for 3.754 and 32^{VI} for .000032; 25 379 for 25.379; 0|25 for .25 and 12|345 for 12.345; and 3_{25} for 3.25.

The use of the decimal point is probably due to John Napier, a Scotchman, who in 1617 published a treatise on decimals in which the point was used in a few instances.

III. Preliminary Work With Decimals in the Grades

92. United States Money. The first work done with decimals in the grades is in connection with United States Money. This is the natural method of introduction in this

*See Section 87.

country as our money system is the only decimal system of units in common use. At least as early as the third grade the pupils learn to write dollars and cents, using the dollar sign and the decimal point. They also learn to add and subtract dollars and cents so written and to multiply and divide them by simple integers. Even in this preliminary work the pupils should use the term *decimal point* and understand the relation between the places on the right of the point and those on the left. Thus in \$3.27, the 3 is dollars, the 2 is dimes and the 7 is cents.

IV. Addition and Subtraction of Decimals

93. Addition and Subtraction. The addition and subtraction of decimals presents no great difficulties. The most important thing is to see that the pupils form the habit of putting down the decimals with the decimal points in a straight line one below the other. The pupils should see why this is necessary, namely that it is the result of the general principle that only numbers having the same name can be added or subtracted. Units can only be added to units, tens to tens, etc., so for convenience in adding and subtracting units are placed under units and tens under tens. With integers this means that the right hand column must be kept straight, with decimals that the decimal points must be kept one under the other in a straight line.

94. Effect of Ciphers at the Right of Decimals. In adding and subtracting decimals the pupils should meet cases in which the decimal in the result ends in one or more ciphers, as in the following.

$$\begin{array}{r} 3.26 \\ 1.54 \\ \hline 4.80 \end{array}$$

In subtracting decimals they should meet cases in which the minuend has fewer decimal places than the subtrahend so

that ciphers must be added to the minuend, at least mentally. Below, in subtracting the pupils must think of the minuend as 5.80.

$$\begin{array}{r} 5.8\,0 \\ 2.14 \\ \hline \end{array}$$

In order to understand cases such as given above the pupils must understand that adding or dropping a cipher at the right of a decimal does not change its value. There are two ways of making this point clear. 5.8 means 5 units and 8 tenths, if a cipher is added at the right it becomes 5 units, 8 tenths *and no hundredths,* so the value is not changed. Or; $.8=\frac{8}{10}$, if a cipher is added at the right it becomes .80 or $\frac{80}{100}$, but $\frac{80}{100}=\frac{8}{10}$, as multiplying the numerator and denominator by the same number does not change the value of the fraction. In the same ways it can be made clear to the pupils that dropping a cipher at the right of a decimal does not change the value. This is left to the student as an exercise.

V. Multiplication and Division

95. Multiplication by an Integer. The method of multiplying a decimal by an integer can be discovered by the pupils in several ways. Three methods are suggested below.

Examples	*Method of Solution*
$\begin{array}{r} .2 \\ 3 \\ \hline \end{array}$	$3\times.2=3\times\frac{2}{10}=\frac{6}{10}=.6$
$\begin{array}{r} .24 \\ 2 \\ \hline \end{array}$	$2\times.24=2\times\frac{24}{100}=\frac{48}{100}=.48$
$\begin{array}{r} 1.21 \\ 4 \\ \hline \end{array}$	$4\times1.21=4\times\frac{121}{100}=\frac{484}{100}=4.84$

This method utilizes the pupil's knowledge of common fractions in order to obtain the results to two or three simple cases. It is then seen that the results could be obtained

directly, without the use of common fractions in the following way.

To multiply a decimal by an integer, multiply as in whole numbers and point off as many decimal places in the product as there are in the multiplicand.

A second method of deriving this rule makes use of the pupil's knowledge of concrete numbers to obtain the results to a few simple cases.

Examples	*Method of Solution*
.3	2×\$3=\$6
2	2×3 tenths=6 tenths
.07	2×\$7=\$14
2	2×7 hundredths=14 hundredths

A third method, somewhat similar to the last follows.

Example	*Method of Solution*
3.23	The pupils think:
3	1. 3×3 hundredths=9 hundredths, so
9.69	there are 9 hundredths in the product.
	2. 3×2 tenths=6 tenths, so there are 6 tenths in the answer. Since it is 6 tenths the decimal point is immediately in front of the 6.
	3. 3×3 units= 9 units, so there are 9 units in the product.

After working one or two examples in this way the rule given above is formulated.

96. Multiplication of Decimals by Powers of 10. The short method of multiplying a decimal by ten can be discovered by actually performing the multiplication in a few cases.

Example	*Method of Solution*
3.27 10 32.70	The pupil multiplies 327 by 10 by annexing a cipher, and points off two decimal places in the product as there are two in the multiplicand. The final cipher may be dropped leaving 32.7.

By comparing the final result 32.7 with the original number 3.27 and by working out other similar cases in the same way the following rule is obtained.

To multiply a decimal by 10, *move the decimal point one place to the right.*

This rule may also be obtained by working out several examples by changing to common fractions.

$$10\times.327=10\times{}^{327}/_{1000}={}^{327}/_{100}=3{}^{27}/_{100}=3.27$$

The pupils should also see *why* moving the point one place to the right multiplies by 10. They should see that moving the point one place to the *right* puts each digit one place further to the *left* and so multiplies the value of each digit and of the whole decimal by 10. Thus in 3.28 if the point is moved one place to the right, as 32.8, the 3 is moved from units to tens place, the 2 is moved from tenths to units, and the 8 from hundredths to tenths.

The method of multiplying by 100 is easily derived from the method of multiplying by 10. As $100=10\times10$, multiplying by 100 is the same as multiplying by 10 and that result by 10. To multiply by ten the point is shifted one place to the right and to multiply that result by 10 it is moved another place to the right, therefore,

To multiply a decimal by 100, *move the decimal point two places to the right.*

The rules for 1000 and the higher powers of 10 can be reasoned out in the same way.

97. Dividing a Decimal by an Integer. In dividing an integer by an integer the pupils have formed the habit of placing the quotient above the dividend and placing the first quotient figure above the last figure of the dividend used in the first division. In this way each figure in the quotient is kept in its proper place, units above units, etc.* Consequently if the dividend is a decimal fraction the decimal point in the quotient will be directly above the decimal point in the dividend. This method should be developed in the grades from simple cases such as the following.

Example. *Method of Solution.*

.23
3)$\overline{.69}$

The decimal point is first put in the quotient above the point in the dividend. The first division is .6÷3. 6¢÷3=2¢, so 6 tenths÷3=2 tenths, or .6÷3=.2. Therefore, 2 is written above the 6 in tenths place. The next division is .09÷3 which in the same way is found to give 3 hundredths so 3 is placed in hundredths place, above the 9.

Other simple examples such as 2)$\overline{4.8}$ and 4)$\overline{.816}$ can be reasoned out in the same way and from such examples the following rule can be formed.

To divide a decimal by an integer, put a decimal point in the quotient above the decimal point in the dividend and divide as with integers.†

98. Dividing Decimals by Power of 10. The rules for dividing decimals by powers of ten can be discovered by dividing a few cases such as the following.

32.87	3.287	.3287
10)$\overline{328.7}$	100)$\overline{328.7}$	1000)$\overline{328.7}$

*See Section 56. †The pupils must be careful to place the first quotient figure above the last figure of the dividend used in the first division. This habit should have been thoroughly established in the division of integers.

The pupils should also understand why these rules work. The explanation is left to the student as an exercise.

99. Multiplying a Decimal by a Decimal. Several methods may be used for discovering the method of multiplying a decimal by a decimal. Several methods follow.

$$.3\times.4=\tfrac{3}{10}\times\tfrac{4}{10}=\tfrac{12}{100}=.12$$

$$.2\times.12=\tfrac{2}{10}\times\tfrac{12}{100}=\tfrac{24}{1000}=.024$$

$$.23\times.03=\tfrac{23}{100}\times\tfrac{3}{100}\times\tfrac{6}{10000}=.0069$$

From simple cases such as those given above the pupils formulate the following rule.

To multiply a decimal by a decimal, multiply as in whole numbers and point off in the product as many decimal places as there are in the multiplier and multiplicand together.

Another method of discovering the above rule follows.

Example	*Method of Solution.*
3.23	To multiply by .3 means to find $\frac{3}{10}$ of the
.3	multiplicand. To find $\frac{3}{10}$ of a number we must
.969	multiply by 3 and divide by 10. $3\times3.23=9.69$ and $9.69\div10=.969$

Several examples may be worked in this way and the rule formulated.

A third method of discovering the rule is given below.

Example.	*Method of Solution.*
22.3	$.2\times.3=\frac{2}{10}\times\frac{3}{10}=\frac{6}{100}=.06$, so 6 is written in
.2	hundredths place. $.2\times2=.4$, so 4 is written in
4.46	tenths place. $.2\times20=4$, so 4 is written in units place.

Again the rule may be discovered by working out several examples in the same way.

100. Dividing by a Decimal. The most satisfactory way of dividing by a decimal is to first make the divisor a whole number by using the principle that both the divisor and dividend may be multiplied by the same number without changing

the quotient.* The division may then be performed as in the division of a decimal by an integer, placing the decimal point in the quotient above the decimal point in the dividend.

Example I	*Explanation.*
3.706÷.17 21.8 17)370.6 34 30 17 136 136	The divisor is made an integer by shifting the point two places to the right (multiplying by 100) and, in order not to change the result, the decimal point in the dividend is also shifted two places to the right. The division is then performed as in dividing a decimal by an integer, being careful to place each quotient figure in its proper place.
Example II	*Method of Solution*
.0017856÷5.2 .000343 52)<u>.017836</u> 156 223 208 156 156	Shift the decimal point one place to the right in both the dividend and divisor (multiply by 10). The first division is 52 into 178, so the first quotient figure is placed above the 8 and the vacant places filled in with ciphers.

In order to avoid writing the numbers twice a caret ^ may be used to mark the position of the decimal points in the dividend and divisor after they have been shifted so as to make the divisor an integer. If this is done the decimal point in the quotient will be over the caret in the dividend. The examples given above are given again below, showing the

*See Section 54.

use of the caret. The pupils should never be permitted to erase the decimal points and put them in their new positions. They should either use the caret to mark the new positions of the points or rewrite the numbers.

```
      21 . 8                .000343
.17^)3.70^6        5.2^).0^017836
      34                    156
      30                    223
      17                    208
     136                    156
     136                    156
```

101. Checking Calculations with Decimals. The four processes with decimals may be checked in the same ways as the corresponding processes with integers. The commonest error is the placing of the decimal point in the wrong place in multiplication and division. The best way to check the location of the decimal point in these processes is to form a rough estimate of the result before carrying out the process in detail and to use this as a check on the final result.

Example I

```
  3.27
   5.7
 2289
1635
18.639
```

Check:

Use 6 and 3 as the nearest easy approximations to 5.7 and 3.27. Then $6\times3=18$, which is the approximate value of the product.

Example II

```
  27.8
   .37
 1946
  834
10.286
```

Check:

$30\times\frac{4}{10}=12$

Example III *Check*:

2.898÷.07 $3 \div 7/100 = 3 \times 100/7 = 300/7 = 43$

41.4

7)289.8

102. Uneven Division. In practical work when a division does not come out even, it is customary to carry it out to enough decimal places to give the desired degree of accuracy. If the remainder is less than half the divisor it is dropped. If the remainder is equal to or greater than half the divisor the last digit of the quotient is increased by 1. In this way the *nearest* digit is used in the last place of the quotient. In the first example given below the result is obtained to the *nearest hundredth,* in the second to the *nearest thousandth.*

Example I.

7.4÷3.14=2.36 (approximate to the nearest hundredth)

2.35

314)740.

628

1120

942

1780

1570

210

Example II.

1.372÷4.3=.319 (approximate to the nearest thousandth)

.319

43)13.72

12 9

82

43

390

387

3

Exercise. The Four Processes with Decimals

1. Explain the effect of dropping a cipher from the right of a decimal. Of adding a cipher.

2. What effect does adding a cipher to the right of an integer have on its value? Why?

3. Explain the effect of moving the decimal point one place to the right. Two places. Three places. One place to the left. Two places. Three places.

4. Develop in detail each of the following as you would in the grades. Use different examples from those used in the text.

1. Multiplication of decimals by integers.
2. Multiplication of decimals by 10, 100 and 1000.
3. Division of decimals by integers.
4. Division of decimals by 10, 100 and 1000.
5. Multiplication of a decimal by a decimal.
6. Division of a decimal by an integer.
7. Division of a decimal by a decimal.

5. Develop rule for multiplying a decimal by an integer by using the meaning of multiplication, namely; $2 \times 41.2 = 41.2 + 41.2$. Develop the rule for multiplying a decimal by 10 in the same way.

6. Divide using the method given in Sections 97 and 100. Carry results to nearest thousandth. Check by estimating result.

1. $7.583 \div 37$	6. $31.576 \div 7.4$	11. $.0197 \div 7.2$
2. $38.52 \div 127$	7. $.51897 \div .0039$	12. $9 \div 3.1416$
3. $.071 \div 32$	8. $857 \div 3.1416$	13. $.000376 \div .41$
4. $1.56 \div 823$	9. $157.2 \div 17.81$	14. $81 \div 959$
5. $50.92 \div 91$	10. $3 \div 876$	15. $107.2 \div 6.283$

VI. Reduction of Decimals and Common Fractions

103. Changing Common Fractions to Decimals. It is often necessary to change common fractions to decimals. If the denominator of the common fraction is some power of ten

the change is simply one of notation or method of writing. Thus $\frac{7}{10}$ and .7; $\frac{19}{100}$ and .19 are simply different methods of writing the same thing.

When the denominator of the common fraction is not a power of ten it can sometimes be changed to a power of ten by multiplying or dividing both the numerator and denominator by the same number.

1. $\frac{3}{5}=\frac{6}{10}=.6$ 3. $\frac{19}{30}=\frac{6\frac{1}{3}}{10}=.6\frac{1}{3}$

2. $\frac{17}{25}=\frac{68}{100}=.68$ 4. $\frac{173}{200}=\frac{86\frac{1}{2}}{100}=.86\frac{1}{2}$

In general to change a common fraction to a decimal it is necessary to carry out the indicated division.

$\frac{17}{23}=.739$ (approximate to the nearest thousandth)

$$
\begin{array}{r}
.739 \\
23\overline{)17.0} \\
16\,1 \\
\hline
90 \\
69 \\
\hline
210 \\
207 \\
\hline
3
\end{array}
$$

The decimal equivalents of the simple common fractions should be memorized.

$\frac{1}{2}=.5$	$\frac{1}{4}=.25$	$\frac{2}{5}=.4$
$\frac{1}{3}=.3\frac{1}{3}=.33\frac{1}{3}$	$\frac{3}{4}=.75$	$\frac{3}{5}=.6$
$\frac{2}{3}=.6\frac{2}{3}=.66\frac{2}{3}$	$\frac{1}{5}=.2$	$\frac{4}{5}=.8$
	$\frac{1}{8}=.12\frac{1}{2}=.125$	

104. Changing Decimals to Common Fractions. In changing to common fractions, the decimal is first written in the form of a common fraction by dropping the decimal point and writing the denominator. It must then be simplified, if complex, and reduced to lowest terms.

$.8 = {}^{8}/_{10} = {}^{4}/_{5}$

$.28 = {}^{28}/_{100} = {}^{7}/_{25}$

$.03\frac{1}{3} = \frac{3\frac{1}{3}}{100} = {}^{1}/_{300} = {}^{1}/_{30}$

The shortest way to simplify the complex fraction $\frac{3\frac{1}{3}}{100}$ is by multiplying both the numerator and denominator by 3.

$.2\frac{1}{2} = \frac{2\frac{1}{2}}{10} = {}^{5}/_{20} = {}^{1}/_{4}$

$.006\frac{1}{4} = \frac{6\frac{1}{4}}{1000} = {}^{25}/_{4000} = {}^{1}/_{160}$

EXERCISE. REDUCTION OF DECIMAL AND COMMON FRACTIONS

1. Change the following to decimals mentally, writing only the equivalent decimal fraction.

1. $^{7}/_{10}$	4. $^{3}/_{30}$	7. $^{757}/_{3000}$
2. $^{12}/_{25}$	5. $^{7}/_{20}$	8. $^{7}/_{40}$
3. $^{21}/_{200}$	6. $^{9}/_{75}$	9. $^{157}/_{300}$

2. Change the following to decimals. Find decimals to nearest hundredth only.

1. $^{7}/_{13}$	4. $^{17}/_{7}$	7. $^{13}/_{17}$
2. $^{238}/_{579}$	5. $^{7}/_{9}$	8. $^{131}/_{237}$
3. $^{7}/_{583}$	6. $^{9}/_{11}$	9. $^{837}/_{799}$

3. Change to common fractions. Simplify and reduce results to lowest terms.

1. .8	5. $.6\frac{2}{3}$	9. .002
2. .6	6. $.22\frac{1}{2}$	10. $.8\frac{1}{3}$
3. .28	7. $.6\frac{1}{4}$	11. 2.05
4. .625	8. $.0\frac{2}{3}$	12. $.00\frac{1}{3}$

VII. CONTRACTION AND EXPANSION OF DECIMALS

105. Contraction. In percentage it is often necessary to change or *contract* a decimal that is written as thousandths or ten thousandths to hundredths. Thus .235=.23½, .5875=.58¾ and .388=.38⅘.

In the .235 the 23 in the first two places after the point is .23. Besides there is 5 in the next place to the right. But the next place to the right has a value $\frac{1}{10}$ as great as hundredths place, so 5 in this place means $\frac{5}{10}$ or $\frac{1}{2}$ of a hundredth. The pupils should be able to make changes of this kind in simple cases such as those given above.

106. Expansion. A complex decimal may often be *expanded* into a simple decimal.

$.12\frac{1}{2} = .125$
$.8\frac{1}{4} = .825$
$.027\frac{3}{5} = .0276$

In the $.12\frac{1}{2}$ the $\frac{1}{2}$ of a hundredth may be written as $\frac{5}{10}$ of a hundredth, but the next place to the right has a value equal to $\frac{1}{10}$ of a hundredth so 5 in this place means $\frac{5}{10}$ or $\frac{1}{2}$ of a hundredth.

EXERCISE. CONTRACTION AND EXPANSION OF DECIMALS

1. Contract the following to hundredths.

1.	.376	3.	.7525	5.	27.897
2.	3.565	4.	.18125	6.	.852

2. Expand the following complex decimals into simple decimals.

1.	$.783\frac{1}{2}$	4.	$2.897\frac{4}{5}$
2.	$3.2\frac{1}{4}$	5.	$.00\frac{3}{4}$
3.	$87.65\frac{1}{8}$	6.	$7.89\frac{9}{25}$

CHAPTER VIII.

THE FUNDAMENTALS OF PERCENTAGE

I. The Meaning and History of Percentage

107. History. The term *per cent* comes from the Latin *per centum* meaning *by the hundred.* Originally per cent meant *by the hundred* or *out of a hundred.* Thus 6 per cent meant 6 out of every hundred, and 10 per cent meant 10 out of every hundred. Percentage as a separate subject did not appear in arithmetic until the end of the sixteenth century, although the practice of reckoning by the hundred originated before that time.

Various methods of writing per cents were used during the sixteenth century by different writers in arithmetic. A few of these were; *pour 100, pro 100, an 100, per cento, per C, per 100, P 100, and P* $\frac{0}{0}$. Our present sign % goes back to the first half of the fifteenth century although it did not come into general use until much later.

108. Present Meaning. Although per cent originally meant out of a hundred it now has a much wider meaning. We now speak not only of 7% and 30% but also of 250% and even of 1000%. According to the original meaning such expressions as 250% and 1000% would be meaningless. *A per cent is a fraction having 100 for a denominator. It is written by writing the numerator followed by the sign % to indicate the denominator 100.*

109. Teaching the Meaning of Per Cent. Percentage is no longer taught as a new subject but simply as a continuation of the work with fractions. The pupils are already familiar with two methods of writing hundredths and must now learn a new name for hundredths and a third way of writing. They are told that $\frac{23}{100}$ or .23 can also be written 23% and that it

is read 23 per cent. The pupils should be encouraged to report any use of the term that they may encounter. The advertisements of sales in the local newspapers should be considered in this connection. They should also be given plenty of practice in writing hundredths in the three different ways such as the following.

1. Write 7% in two other ways.
2. Write $17/100$ in two other ways.
3. Write .23 in two other ways.
4. Write *81 hundredths* in three ways.

From practice of this kind the pupils learn that *per cent* and *hundredths* are two different names for the same thing, that any number of hundredths is the same number of per cent, that any number of per cent is the same number of hundredths, and that hundredths may be written in three different ways.

Exercise. Writing Hundredths

1. Write each of the following in three ways.

1. 7 hundredths
2. 34 hundredths
3. 237 hundredths
4. $8\frac{1}{3}$ hundredths
5. $\frac{1}{2}$ of a hundredth
6. 5376 hundredths

2. Write each of the following in two other ways.

1. 18%
2. 237%
3. $7\frac{1}{2}$%
4. $83/100$
5. $72/100$
6. $7/100$
7. .41
8. .07
9. 3.24

II. Reductions

110. General Discussion. In order to base the work in percentage on the work already done with common and decimal fractions, it is necessary that the pupils be able to translate per cents into common and decimal fractions, and common and decimal fractions into per cents. Much practice must be

given. The changes should be based directly upon the fact that per cent means hundredths and upon the three methods of writing hundredths discussed above.

111. Changing Per Cents to Decimals. Since per cent is only another name for hundredths, the changing of a per cent to a decimal involves no change except in the name and method of writing. Thus 27% means 27 hundredths which written as a decimal is .27. 12½% means 12½ hundredths which written as a decimal is .12½ or .125.

To change a per cent to a decimal, think of it as hundredths and write it as a decimal.

112. Changing Per Cents to Common Fractions. Again the changing of a per cent to a common fraction involves nothing but a change of name and a change in the method of writing. 18% means 18 hundredths which written as a common fraction is $^{18}/_{100}$ or $^{9}/_{50}$. 8⅓% means 8⅓ hundredths which written as a common fraction is $\frac{8\frac{1}{3}}{100}=\frac{25}{300}=\frac{1}{12}$.

To change a per cent to a common fraction, think of it as hundredths and write it as a common fraction, simplify and reduce to lowest terms.

113. Changing Decimals to Per Cents. If a decimal is expressed as hundredths as .27 or .83 it can be changed to a per cent by simply changing the name and the method of writing, as .27=27%, .83=83%. If the decimal is not expressed as hundredths, it must first be changed to hundredths by contracting or expanding, as .7 equals 70 hundredths or 70%, .325 equals 32.5 hundredths or 32.5%.

To change a decimal to a per cent first read the decimal as hundredths and write it as the same number of per cent.

Any decimal may be read as hundredths by reading the digits up to and including hundredths place as that many hundredths, and the remaining digits as a decimal fractional part of a hundredth. Thus .273 is 27 hundredths and .3 of a

hundredth, or 27.3 hundredths, or 27.3%. 2.3875 is 238 hundredths and .75 or ¾ of a hundredth, or 238¾ hundredths, or 238¾%.

114. Changing Common Fractions to Per Cents. Sometimes common fractions can be easily changed to per cents by first changing to equivalent common fractions having 100 for a denominator and then rewriting as a %. Thus $\frac{8}{25}=\frac{32}{100}=32\%$, $\frac{17}{300}=\frac{5\frac{2}{3}}{100}=5\frac{2}{3}\%$.

To change a common fraction to a per cent, multiply or divide both the numerator and denominator by some number that will make the denominator 100. Then rewrite as a per cent.

In general, when the denominator can not be changed to 100 by a simple multiplication or division, it is easier to first change the common fraction to a decimal by dividing the numerator by the denominator. When the division does not come out even at hundredths place, it is customary to carry it to the nearest thousandth, that is to the nearest tenth of a per cent. If a higher degree of accuracy is desired the division is carried to the nearest ten thousandth giving the per cent to the nearest hundredth of a per cent.

To change a common fraction to a per cent divide the numerator by the denominator and then change the resulting decimal to a per cent.

Examples	*Method of Solution*
$\frac{7}{13}=53.8\%$ (to the nearest tenth of a per cent)	$13\overline{)7.0}$ = .538 6 5 50 39 110 104 6

$24/79 = 30.38\%$ (to the nearest hundredth of a per cent)

$$
\begin{array}{r}
.3037 \\
79\overline{)24.0} \\
23\,7 \\
\hline
300 \\
237 \\
\hline
300 \\
237 \\
\hline
630 \\
553 \\
\hline
77
\end{array}
$$

115. Equivalents to be Memorized. The per cent equivalents of the simple common fractions that occur frequently in practical calculations should be memorized. The pupils should be able to give instantly the per cent equivalents of all the common fractions given in the following table, and should also be able to give the fractional equivalents of the per cents.

Table of Equivalents.

$\frac{1}{2}=50\%$	$\frac{1}{5}=20\%$	$\frac{2}{5}=40\%$	$\frac{5}{8}=62\frac{1}{2}\%$
$\frac{1}{3}=33\frac{1}{3}\%$	$\frac{1}{6}=16\frac{2}{3}\%$	$\frac{3}{5}=60\%$	$\frac{7}{8}=87\frac{1}{2}\%$
$\frac{2}{3}=66\frac{2}{3}\%$	$\frac{1}{8}=12\frac{1}{2}\%$	$\frac{4}{5}=80\%$	$\frac{3}{10}=30\%$
$\frac{1}{4}=25\%$	$\frac{1}{10}=10\%$	$\frac{5}{6}=83\frac{1}{3}\%$	$\frac{7}{10}=70\%$
$\frac{3}{4}=75\%$	$\frac{1}{12}=8\frac{1}{3}\%$	$\frac{3}{8}=37\frac{1}{2}\%$	$\frac{9}{10}=90\%$

The equivalents given in the first two columns are the most important. A knowledge of the others is convenient but not absolutely necessary.

Exercise. Reductions

1. Practice giving the per cent equivalents of the following common fractions. Vary the order by first giving them by columns, then by rows, etc. Do not write the answers.

$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{12}$	$\frac{5}{8}$	$\frac{3}{10}$
$\frac{1}{6}$	$\frac{1}{10}$	$\frac{7}{8}$	$\frac{3}{5}$	$\frac{1}{3}$
$\frac{4}{5}$	$\frac{1}{8}$	$\frac{9}{10}$	$\frac{1}{4}$	$\frac{3}{8}$
$\frac{7}{10}$	$\frac{2}{5}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{5}$

2. Practice giving the common fraction equivalents of the following per cents. Vary the order as in Exercise 1. Do not write answers.

33⅓%	20%	30%	70%	37½%
60%	66⅔%	62½%	80%	25%
87½%	83⅓%	8⅓%	16⅔%	90%
10%	40%	75%	50%	12½%

3. Think of the following per cents as hundredths and write as decimals.

1. 5%	7. 6⅔%	13. 120%	19. .3%
2. 43%	8. 57½%	14. ⅝%	20. 95.2%
3. 65%	9. 18¼%	15. 8⅗%	21. 250%
4. 3897%	10. 277%	16. 300%	22. 205%
5. 73.4%	11. 125%	17. ½%	23. ¾%
6. 32%	12. 3½%	18. 2.7%	24. 1⅓%

4. Think of the following per cents as hundredths and write as common fractions. Simplify and reduce to lowest terms.

1. 15%	6. 57½%	11. ⅓%	16. ⅔%
2. 28%	7. 37%	12. 2760%	17. 32½%
3. 7½%	8. 220%	13. 238%	18. 178.4%
4. 3⅓%	9. 85%	14. ¾%	19. 6⅔%
5. ½%	10. 375%	15. 72%	20. 8%

5. Read the following decimals as hundredths and write as per cents.

1. .05	6. .3½	11. 3.025	16. 3.1
2. .17	7. 4.5	12. .908	17. 37.20
3. .2	8. .8¼	13. .00⅓	18. .0175
4. 3.27	9. .1725	14. .7¾	19. .0025
5. .625	10. .8075	15. 2.027	20. .0005

6. Mentally change the following common fractions to equivalent fractions having 100 for denominator and write as per cents.

1. $\frac{9}{10}$	4. $\frac{17}{25}$	7. $3\frac{1}{2}$
2. $\frac{14}{300}$	5. $5\frac{1}{4}$	8. $\frac{183}{700}$
3. $\frac{9}{20}$	6. $\frac{14}{40}$	9. $\frac{43}{50}$
		10. $\frac{581}{700}$
		11. $\frac{9}{75}$
		12. $\frac{13}{40}$

7. Change the following common fractions to per cents by dividing the numerator by the denominator. Find the per cents to the nearest tenth of a per cent.

1. $\frac{5}{9}$	4. $\frac{27}{73}$	7. $\frac{549}{473}$	10. $\frac{385}{8}$
2. $\frac{8}{11}$	5. $\frac{897}{976}$	8. $\frac{7}{986}$	
3. $\frac{9}{17}$	6. $\frac{18}{7}$	9. $\frac{6}{7897}$	

III. The Three Problems of Percentage

116. General Statement. Three types of problems occur in percentage. The pupils should see that these problems are not new but are only problems in fractions, with which they are already familiar, stated in a new way. The method of handling the percentage problem should be the same as the method of handling the corresponding problem in fractions. The three problems of percentage are stated below in such a way as to bring out the relation to the corresponding problems in fractions.

1. To find any per cent (fractional part) of any number.

2. To compare two numbers and find what per cent (fractional part) the one is of the other.

3. To find the whole number when some per cent (fractional part) of it is given or known.

117. Finding Per Cents of Numbers. The problem of finding a per cent of a number is the old problem of finding a fractional part of a number stated in a new way.

Examples	*Method of Solution*
1. 25% of \$18.75	$\frac{1}{4} \times \$18.75 = \$4.69$
2. 17½% of \$32.18=\$5.63	4.69 32.18 .175 16090 22526 3218 5.63150

To find any per cent of any number, change the per cent to either a common or decimal fraction and multiply.

118. Finding What Per Cent One Number is of Another. The problem of finding what per cent one number is of another number is the old problem of comparing numbers, or of finding the ratio of one number to another. The pupils should have had plenty of practice in comparing numbers by finding the ratio of the one to the other, before taking up the work with percentage.

To find what per cent one number is of another, first compare the one number to the other and express the comparison as a ratio or common fraction. Then change the common fraction to a per cent.

In comparing numbers and expressing the ratio of the one to the other, the commonest error is to write the ratio or common fraction upside down. The ratio or fraction ⅞ expresses the ratio of 7 to 8, namely that 7 is ⅞ of 8. The number that is the base of the comparison, or to which the other number is being compared is always the denominator of the fraction.

Examples	*Method of Solution*
1. 17 is what per cent of 83?	17 is to be compared to 83, so the ratio is $\frac{17}{83}$ which equals 20.5%.

2. What per cent of 398 is 873? 873 is to be compared to 398, so the ratio is $\frac{873}{398}$, which equals 219.3%.

119. To Find the Whole Number When Some Per Cent of it is Known. The problem of finding the whole number when some per cent of it is known is the old problem of finding the whole when some fractional part of it is known.

1. 75% N=36

$\frac{3}{4}$ N=36

N=36÷$\frac{3}{4}$=$\frac{36}{1}\times\frac{4}{3}$=48

2. 97.2% S=$18.50

.972 S=$18.50

S= 18.50÷.972=$19.03

```
        19.03
   972)18500.
       972
      8780
      8748
       3200
       2916
        284
```

To find the whole number when some per cent of it is known, divide the known part by the per cent changed to a common or decimal fraction.

The explanation of the above method is obvious to any one who is familiar with algebra as it involves only the solution of a simple linear equation. In schools in which algebra is not introduced in the upper grades the method can be explained as follows.

32% N=76

.32 N=76

76 is the product of two factors, the unknown number N and .32. As the product and one factor are known the other factor (N) can be found by dividing the product (76) by the known factor (.32). N=76÷.32.

120. The Fundamentals of Percentage. Success in percentage depends upon the ability to change per cents to common and decimal fractions, to change common and decimal fractions to per cents, to find per cents of numbers, to find what per cent one number is of another, and to find the whole number when some per cent of it is known. These seven constitute the fundamentals of percentage and must be mechanized through drill. If this is not done successful work is impossible.

EXERCISE. THE FUNDAMENTALS OF PERCENTAGE

1. Find the following mentally.

1. 66⅔% of 27
2. 20% of 75
3. 12½% of 72
4. 25% of 184
5. 33⅓ of 171
6. 7% of 300
7. 16⅔% of 54
8. 10% of 78
9. 75% of 48
10. 8⅓% of 60
11. 50% of 17
12. 1% of 375
13. ½% of 800
14. ⅓% of 960
15. 200% of 37
16. 250% of 8
17. 300% of 5
18. 725% of 8
19. 1000% of 87.23
20. 333⅓% of 9
21. 600% of 32
22. 475% of 12
23. 3000% of 7
24. 2500% of 8

2. Find the following, using pencil and paper if necessary.

1. 12% of 78
2. 7% of 8.75
3. 13½% of $70.25
4. 35% of 82
5. 9% of 72
6. 25% of 7815
7. ½% of $86.05
8. 372% of 87
9. 7⅓% of 972
10. 87.3% of $17.50
11. 79.9% of 832
12. 387.2% of 76
13. .8% of 756
14. .03% of 931

3. Compare the number in column A to the number opposite it in column B. Express the comparison as a common fraction and then as a per cent. Do not use pencil or paper.

A	B	A	B
18	36	2	24
8	48	21	14
17	51	17	17
3	8	18	6
7	56	25	3
9	45	48	32
24	32	27	36
15	45	6	6
24	12	350	70

4. In Exercise 3 compare the number in column B to that opposite it in column A. Express the comparison as a common fraction and then as a per cent.

5. Compare the number in column A to that opposite it in column B. Express the comparison as a common fraction and then change to a per cent. Figure per cents to nearest tenth of a per cent. Use pencil and paper if necessary.

A	B	A	B
17	52	18	7
87	395	787	159
12½	17	5	987
⅔	5/7	1	323
876	952	7875	89

6. In Exercise 5 find what per cent the number in column B is of the number opposite it in column A.

7. Find the whole number N, in each of the following cases.

1. 8% N=$72.07
2. 75% N=891
3. 17% N=75
4. 57½% N=$18.75
5. 70% N=105
6. 12½% N=392
7. 37.7% N=$87
8. 235% N=$125
9. .9% N=$9.13
10. 132.3% N=$67.19

IV. General Applications of Percentage

121. General Statement. There are many important applications of percentage in addition to the more or less technical business applications which are discussed in the following chapters. Among the most important of these are (1) Problems involving the per cent of various constituents in a whole, (2) Problems involving per cent of increase or decrease of some variable quantity, and (3) Problems involving the comparison of two quantities. The per cent of increase or decrease of a quantity means the per cent the amount of the increase or decrease is of the original value of the quantity. Thus if the original value is 8 and the new value 10, the amount of increase is 2, and this increase is $\frac{2}{8}$ or 25% of the original value.

Problems. General Application of Percentage

Note. **The student should form the habit of checking in some way the results obtained in the solution of a problem, whenever it is possible to do so.**

1. A farmer weighs his corn in the fall and finds that he has 85,750 lbs. How many bushels will he have in the spring if it loses 20% of its weight through drying? (70 pounds=1 bu.)

2. The fodder from one acre of corn weighed 40,000 lbs. What is the weight of this fodder when dry if it loses 62½% of its weight in drying?

3. A man's yearly income is $2500. He spends 25% of his income for food, 20% for rent, 20% for clothes, 15% for operating expenses, 10% for recreation, church, books, etc., and saves 10%. Find the amount spent for each item and the amount saved. How can you check the accuracy of these results?

4. A salesman for a wholesale house works on a commission of 5% of his sales. On sales in excess of $100,000 per year he gets an additional commission of 1%. His total sales for 1920

amounted to $161,872.39 and his expenses for the same year were $4,150.81. Find his total commissions and net income above expenses.

5. A bank with deposits of $875,642 was closed by the bank examiner. Assets of $751,387 were found. What per cent of his money did each depositor receive?

Note. Figure all per cents to the nearest tenth of a per cent. As only three figures at most are wanted in the result it will be sufficiently accurate when figuring per cents involving large numbers to use only the first four figures of the data, using the nearest figure in the fourth place. In this problem use $875,600 for the $875,642 and $751,400 for the $751,387. Much work can be saved by using data only to the degree of accuracy desired in the results. Work this problem using the exact data given, then work it using the approximations suggested and compare results. (See Section 152.)

6. A merchant who failed in business was found to have assets of $78,850 and liabilities of $110,730. What per cent of his indebtedness can he pay? How much will a creditor receive to whom the merchant owes a bill of $7,895?

7. In a certain school room there were 24 pupils enrolled. During one school month of 20 school days, pupils were absent a total of 71 half days. Find the per cent of attendance.

8. A certain fourth grade is composed of 14 boys and 17 girls. The girls constitute what per cent of the total enrollment? The boys? Check.

9. Water expands about 10% of its volume when it freezes. How many cubic feet of water are needed to make 10 cu. ft. of ice. Check.

10. There were 66,810,836 head of cattle on the farms of the United States at the end of 1919. Of these 120,540 were pure bred. What per cent of the total number were pure bred?

11. When limestone weathers, 96% may be dissolved and carried away. How many cubic feet of limestone would make 12 cubic feet of soil? Check.

12. The area of the windows of a school room should be at least 20% of the area of the floor. If a school room is 35′

by 24.5′ and has 6 windows each 10′ by 4′, find what per cent the window area is of the floor area. Is the window area sufficient?

13. Find what per cent the window area is of the floor area in your school room, taking whatever measurements are necessary.

14. In the Bowling Green State Normal College, A the highest mark, is supposed to be given to about 8% of the students, B to 25%, C to 34%, D to 25% and F (Failure) to 8%. If an instructor has altogether 72 students in a given subject about how many of each mark would he expect to give if the classes are of average ability? Check.

15. In a certain high school, an English teacher gave grades for one semester as follows: 5 A's, 27 B's, 60 C's and 41 F's (Failure). A teacher of Commercial Geography and Industrial History in the same school gave 56 A's, 32 B's, 12 C's and 8 F's. To what per cent of her pupils did each teacher give a mark of A? Of B? Of C? Of F? What do you think of the grades given by these two teachers? Check results.

16. If your school uses a letter system of grading find out from your instructor how many students received each mark on the last test. Find what per cent of the class received each mark.

17. The population of Toledo, Ohio in 1910 was 168,497, in 1920 it was 243,164. The population of Indianapolis, Indiana in 1910 was 233,650, in 1920 it was 314,194. Find the per cent of increase in population for each city. Which city grew more rapidly during the ten years? How does the per cent of increase give you more information than the actual amount of the increase? Check results.

18. The population of Chicago from 1860 to 1920 is given below.

1860	1870	1880	1890	1900	1910	1920
109,260	298,977	503,185	1,099,850	1,698,575	2,185,283	2,701,705

Find the per cent of increase in population for each 10 year period. During which period did Chicago grow most rapidly?

19. In 1919-20 there were 9523 students attending the state supported colleges and universities in Ohio. Five years before there were only 7492. Find the per cent of increase.

20. In 1919-20 the state of Ohio invested in its colleges $2,322,205. Five years before it invested $1,856,218. Find the per cent of increase. Which increased most in the five years, the attendance or the money invested? (See preceding problem).

21. A certain dye sold for $0.10 an ounce before the war. During the war it sold for $1.37 an ounce. Find the per cent of increase. Check.

22. A coal company recently announced the following reductions in prices.

Pocahontas lump from $10.50 to $9.90 a ton

Peerless Pomeroy lump from $8.75 to $7.90 a ton

Find the per cent of decrease in price for each kind of coal.

23. A farmer shipped a load of hogs to market. The total weight before shipping was 18,970 lbs. On reaching their destination the total weight was 18,720 lbs. Find the per cent of shrinkage in weight. Check.

24. Four steers weighing 1207 lbs., 1183 lbs., 1072 lbs. and 1281 lbs., were shipped by a farmer to the stock yards. The shrinkage in weight due to the transportation was 3.1%. Find the total loss in weight.

25. The population of Akron, Ohio in 1910 was 69,067, in 1920 it was 208,435. Find the per cent of increase in population.

26. Fill in the last column of the following table.

	Total number of children 14 and 15 years of age	Number Attending School	Per cent Attending School
United States..	3,907,710	3,124,120	.....
Ohio	185,015	162,380	.....
New York.....	325,678	265,353	.....
Illinois	217,579	171,810	.....
Louisiana	81,865	53,736	.....

Which state has the largest percentage of its children of 14 and 15 years of age attending school? Which the smallest? Which have a larger percentage attending than the United States as a whole? Which have a smaller percentage?

27. Fill in the last column of the following table.

	Population, 10 years of age or over. 1920	Number of Illiterates. 1920	Per cent of Illiterates
United States	82,739,315	4,931,905	.....
Ohio	4,624,456	131,006	.....
South Carolina ...	1,219,316	220,667	.....

How does the per cent of illiterates give you more information than the actual number?

28. The population of the United States, 10 years of age or over, in 1910 was 71,638,480 and the number of illiterates was 5,516,163. Did illiteracy increase or decrease in the United States from 1910 to 1920? (See preceding problem.)

29. Complete the following table.

Population United States

Year	In towns of 2500 or more	Rural	Per cent of total population in towns of 2500 or more
1900	30,797,185	45,197,390	.....
1910	42,166,120	49,806,146	.....
1920	54,304,603	51,406,017	.....

What do these figures show?

30. In Ohio 673,858 families live in rented homes, 286,293 families in homes partly paid for, and 432,804 families in homes entirely paid for. Find what per cent of the families in Ohio live in rented homes, in homes partly paid for and in homes entirely paid for? Check.

31. There were 41,609,192 persons in the United States in 1920 who were engaged in gainful occupations. Of these 26.3% were engaged in agriculture, 2.6% in mining, 30.8% in manufacturing and mechanical industries, 7.4% in transportation, 10.2% in trade, 1.9% in public service, 5.2% in professional service, 8.2% in domestic service and 7.5% clerical occupations. Find the number of persons engaged in each line of work. Check.

32. A certain breakfast food loses about 8% of its weight after standing a few weeks. How much will the manufacturer have to put in a box so that the contents will weigh 16 ounces after standing? Check.

33. A student in the eighth grade takes a standard test in arithmetic and makes a score of 5 examples correct out of 7 attempted. Find his per cent of accuracy. Three months later he takes the same test again and makes a score of 9 correct out of 11 attempted. Find his per cent of accuracy on the second test. Also find the per cents of increase in accuracy and in speed.

34. On a certain day of the baseball season the standing of the two leading teams in the National League was as follows:

	Won	Lost
New York....	52	28
Boston	53	30

Find what per cent of total games played was won by each team. Which team was ahead? In the afternoon the two teams played each other and Boston won. Find the per cent of games won by each. Which was ahead?

35. The final standing of the teams in the Northwestern Ohio College Conference at the end of the baseball season of 1921 was as follows. Complete the table.

Team	Games Played	Won	Lost	Tied	Per cent of Games Won
Bowling Green	9	6	1	2	.....
Bluffton	8	6	2	..	.....
Toledo	9	2	5	2	.....
Findlay	8	0	8	..	.....
Defiance	8	5	3	..	.....

36. How many games did your school team win and lose last season? Figure the per cent of games won. If your team plays in a league or conference, figure the per cent of games won by each team in the league and determine their relative standing.

37. A commission merchant sold a car load of melons (1450) for a farmer at 61 cents each, charging a commission of 5% of the amount of the sale. Find the merchant's commission. What did the farmer receive for the melons if freight charges amounted to $20?

38. Using the data of the following table, find the per cent of increase in height and in weight for both boys and girls during each of the 5 year periods.

Average Heights and Weights of Children

Age	Boys		Girls	
	Height (Inches)	Weight (Pounds)	Height (Inches)	Weight (Pounds)
5 years.....	41.6	41.1	41.3	39.7
10 years.....	51.7	65.3	51.3	62.4
15 years.....	62.3	107.1	61.1	106.1

39. If ordinary grass seed costs 10 cents a pound and is 20% weed seed, what is the real cost per pound of the grass seed? Would it be cheaper to buy a clean seed at 12 cents a pound?

40. A farmer sends to a creamery during one month 8617 pounds of milk, testing 4.1% butter fat. If he was paid at the rate of 33¢ per pound for the butter fat, what did he receive for the milk?

41. If the milk is sold for the butter fat, as in the preceding problem, which is more profitable, a cow producing 5387 lbs. of milk a year, testing 3.4%, or one producing 4792 lbs., testing 4.5%? When butter fat is selling at the rate given in the preceding problem, what is the difference in the value of the milk produced by the two cows?

42. One pound of butter fat will make 1⅙ pounds of butter. Supposing that the separator takes out 96% of the butter fat, how many pounds of butter can be made from 500 pounds of milk that tests 4% butter fat?

43. Under the conditions stated in the preceding problem, how many pounds of milk testing 3.9% butter fat will it take to make 12 lbs. of butter? Check.

44. Last year 90% seed corn could be bought at $5 a bushel, 70% seed corn at $4 a bushel. Which was the cheapest and by how much?

45. A certain dress goods shrinks 9% of its length when washed. How many yards would you need to buy of the unshrunk goods if you want 5 yards after shrinking? Check.

46. A housewife uses boiled ham for sandwiches. She finds that she can buy the ham ready boiled and sliced at 60¢ a pound, or can buy a whole raw ham at 38¢ a pound. If the raw ham loses 45% of its weight in cooking and waste and the cost of gas for cooking amounts to 2½¢ for each pound of raw ham cooked, which would be the most economical purchase? What is the difference?

CHAPTER IX.

DENOMINATE NUMBERS

I. Kinds of Numbers

122. Definitions. A number that does not refer to any particular class of object or standard unit of measure is called an *abstract number.* Thus 7, 23, 3¼ and 2.5 are abstract numbers.

Strictly speaking all numbers are abstract, the number in 7 pounds being 7, the pounds simply naming the unit. It is convenient, however, to call numbers such as 7 pounds, $8, and 3 apples, which refer to some particular class of objects or to some standard unit of measure, *concrete numbers.*

All calculations should be performed with abstract numbers, the results only being named or considered as concrete. This is the practice in business and should be followed in the schools.

A concrete number that is expressed in terms of some standard unit or units of measure is called a *denominate* number. Thus 7 lbs., 8 ft. 3 in., and $4 are denominate numbers. A denominate number expressed in terms of a single unit as 25 feet, 8 hours, etc., is called a *simple denominate number.* A denominate number expressed in two or more units such as 8 lbs. 3 oz., or 5 hrs. 20 min., is called a *compound denominate number,* or simply a *compound number.*

123. Practical Use of Compound Numbers. In practical work, quantities are often expressed in two or three units but very seldom in more. Thus numbers such as 3 feet 7 inches and 70° 48′ 13″ are of rather common occurrence but quantities such as 6 miles 120 rods 3 yards 2 feet 2 inches are not met with in practical work. Compound numbers are becoming less and less important in practical affairs owing to the

rapidly growing tendency to express quantities in a single unit and decimal fractional parts of that unit rather than in two or more units. Thus instead of expressing an angle as 27° 15′ it is expressed as 27°.25, instead of expressing a distance as 3 feet 9 inches it is expressed as 3.75 feet. Compound numbers involving two or three units are still of sufficient practical importance, however, to make it necessary that pupils learn in the elementary schools how to work with such numbers. Tables of denominate numbers are given in the appendix for reference.

II. Reductions

124. Definitions. In working with denominate numbers it is often necessary to change the unit or units in which they are expressed. Such change of units is usually called *reduction*. The change may be either to larger or to smaller units.

125. Reduction to Smaller Units. The reduction of denominate numbers to smaller units is shown in the following examples.

Example. Reduce 8 ft. 5 in. to inches.

Solution.

$$\begin{array}{r} 12 \\ 8 \\ \hline 96 \\ 5 \\ \hline 101 \text{ inches} \end{array}$$

Explanation. Since there are 12 inches in 1 foot, in 8 feet there are 8×12 inches$=$96 inches. 96 inches$+5$ inches$=$101 inches.

Example. Reduce 4.7 miles to miles and feet.

Solution.

$$\begin{array}{r} 5280 \\ .7 \\ \hline 3696.0 \end{array}$$

Explanation. One mile equals 5280 feet so .7 mile equals $.7 \times 5280$ ft.$=$3696 ft.

4.7 miles$=$4 miles 3696 feet.

Example. Reduce $\frac{7}{8}$ of a bushel to smaller units.

Solution.

$\frac{7}{8}\times 4=3\frac{1}{2}$

$\frac{1}{2}\times 8=4$

Explanation. One bushel equals 4 pecks. $\frac{7}{8}$ bu.$=\frac{7}{8}\times 4$ pks.$=3\frac{1}{2}$ pks. One peck equals 8 quarts. $\frac{1}{2}$ peck$=\frac{1}{2}\times 8$ quarts $=4$ qts.

$\frac{7}{8}$ bu.$=3$ pks. 4 qts.

126. Reduction to Larger Units. The reduction of denominate numbers to larger units is shown in the following examples.

Example. Reduce 589 sq. in. to sq. ft.

Solution.

$$\begin{array}{r} 4.09 \\ 144\overline{)589.} \\ 576 \\ \hline 1300 \\ 1296 \\ \hline 4 \end{array}$$

Explanation. One square foot contains 144 sq. in. so there are as many square feet in 589 sq. in. as 144 sq. in. is contained in 589 sq. in. or $589\div 144=4\frac{13}{144}=4.09$ (approx.)

589 sq. in.$=4\frac{13}{144}$ sq. ft.$=4.09$ sq. ft. (approx.)

Example. Reduce 352 feet to yards and feet.

Solution.

$$\begin{array}{r} 117 \text{ Rem. } 1 \\ 3\overline{)352} \end{array}$$

Explanation. Since 1 yard equals 3 feet, there are as many yards in 352 ft. as 3 ft. is contained in 352 ft. or $352\div 3=117$ yds. with a remainder of 1 ft.

352 ft.$=117$ yds. 1 ft.

Example. Reduce 3 ft. 5 in. to feet.

Solution.

$$\begin{array}{r} .41 \\ 12\overline{)5.0} \\ 48 \\ \hline 20 \\ 12 \\ \hline 8 \end{array}$$

Explanation. Since there are 12 inches in 1 ft., 5 in. equals $\frac{5}{12}$ of a foot.

3 ft. 5 in.$=3\frac{5}{12}$ ft.$=3.42$ ft. (approx.)

Example. Reduce 18° 27′ 13″ to degrees.

Solution.

27		.45
60	3600)1633.0	
1620		1440 0
13		193 00
1633		180 00
		13 00

Explanation. First reduce 27′ to seconds and add the 13″. This gives 18° 1633″. Since there are 3600 seconds in 1°, 1633″ equals $^{1633}/_{3600}$ degrees, or .45 degrees. (approx.)

18° 27′ 13″=18°.45 (approx.)

III. Addition and Subtraction

127. General Discussion. The processes of addition and subtraction of compound numbers are identical in principle with the addition and subtraction of abstract numbers. The fundamental principle involved is that only like numbers can be added or subtracted.

128. Addition. The process of adding compound numbers is shown in the following example.

Example. Add 5 yd. 2 ft. 3 in., 1 yd. 1 ft. 8 in., and 3 yd. 1 ft. 9 in.

Solution.

5 yd. 2 ft. 3 in.	
1 yd. 1 ft. 8 in.	
3 yd. 1 ft. 9 in.	
10 yd. 2 ft. 8 in.	

Explanation. Place inches under inches, feet under feet and yards under yards. The sum of the first column is 20 in. But 20 inches= 1 ft. 8 in., so place 8 in. under first column in sum and add the 1 ft. to the next column. The sum of the second column is 5 ft., but 5 ft.=1 yd. 2 ft., so the 2 ft. is written in sum under second column and the 1 yd. is added to the next column giving 10 yd.

129. Subtraction. The subtraction of compound numbers is performed in the same way as the subtraction of integers and mixed numbers and any of the methods given in Section 41 for subtracting integers may be used. The greatest diffi-

culty, as in the case of integers, comes from a number in the minuend that is smaller than the corresponding number in the subtrahend. This difficulty can be overcome either by Borrowing or by the method of Equal Additions.* In performing the subtraction the pupils can think in terms of either the Take Away or Additive methods. The pupils should use the same method in subtracting compound numbers that they use in subtracting integers and mixed numbers. An example is given below showing both the Borrowing and Equal Additions methods as applied to compound numbers.

Example. Subtract 2 yd. 2 ft. 7 in. from 7 yd. 2 ft. 3 in.

Solution.

7 yd. 2 ft. 3 in.
2 yd. 2 ft. 7 in.
4 yd. 2 ft. 8 in.

Explanation Using Borrowing. As 3 in. is less than 7 in., 1 ft. is taken from the 2 ft. leaving 1 ft. 1 ft.=12 inches, and 12 in.+3 in.=15 inches. We now have 15 inches in the minuend and 7 inches in the subtrahend so there will be 8 inches in the result. As one of the 2 ft. in the minuend has been used there is only 1 ft. left, which is less than the 2 ft. in the subtrahend, so 1 yd. is taken from the 7 yd. 1 yd.=3 ft. 3 ft.+1 ft.=4 ft. We now have 4 ft. in the minuend and 2 ft. in the subtrahend so there will be 2 ft. in the result. Since one of the 7 yd. in the minuend has been used there are now 6 yd. in the minuend and 2 yd. in the subtrahend so there will be 4 yd. in the result.

Explanation Using Equal Additions. As 3 inches is less than 7 inches, we add one unit of the next higher order, namely 1 ft. or 12 inches to the 3 inches making 15 inches. We now have 15 inches in minuend and 7 inches in subtrahend so there will be 8 inches in the result. Having added 12 inches or 1 ft. to the minuend, we add 1 ft. to the subtrahend giving 3 ft. Since 2 ft. in the minuend is less than the

*See Section 41.

3 ft. we now have in the subtrahend, we add 1 yd. or 3 ft. to the minuend making 5 ft. We now have 5 ft. in minuend and 3 ft. in subtrahend so there will be 2 ft. in the result. Having added 3 ft. or 1 yd. to the minuend, we add 1 yd. to the subtrahend making 3 yd. As there are now 7 yd. in the minuend and 3 yd. in subtrahend there will be 4 yd. in the result.

IV. Multiplication and Division

130. Multiplication. The multiplication of a compound number by a simple abstract number is shown in the following example.

Example. $3\times 18°\ 27'\ 31''$.

Solution.	*Explanation.* $3\times 31''=93''=1'\ 33''$. Write
$18°\ 27'\ 31''$	the $33''$. $3\times 27'=81'=1°\ 21'$. $21'+1'=22'$.
$\quad\quad 3$	Write the $22'$. $3\times 18°=54°$. $54°+1°=55°$.
$55°\ 22'\ 33''$	Write the $55°$.

It is sometimes necessary to find the product of two quantities both of which are expressed in two or more units. In such cases both quantities must first be reduced to a single unit.

Example. Find the area of a rectangle 7 ft. 8 in. by 2 ft. 4 in.

First Solution.

7 ft. 8 in.$=7\frac{2}{3}$ ft.$=\frac{23}{3}$ ft.

2 ft. 4 in.$=2\frac{1}{3}$ ft.$=\frac{7}{3}$ ft.

Area$=\frac{23}{3}\times\frac{7}{3}=\frac{161}{9}=17\ \frac{8}{9}$ sq. ft.

Second Solution.

7 ft. 8 in.$=92$ in.

2 ft. 4 in.$=28$ in.

Area$=28\times 92=2576$ sq. in.$=17\frac{8}{9}$ sq. ft.

131. Division. The division of a compound number by a simple abstract number is shown in the following example.

Example. A certain angle is 37° 17′ 9″. Find half of the angle.

Solution.

$$2\overline{)37^\circ\ 17'\ 9''}\quad 18^\circ\ 38'\ 34''.5$$

Explanation. $37^\circ \div 2 = 18^\circ$ with a remainder of 1° or $60'$. $60' + 17' = 77'$. $77' \div 2 = 38'$ with a remainder of $1'$ or $60''$. $60'' + 9'' = 69''$. $69'' \div 2 = 34''.5$.

It is sometimes necessary to divide one compound number by another of the same kind. In such cases both must be reduced to the same single unit. The process is shown in the following example.

Example. How many pieces 1 ft. 3 in. long can be cut from a piece of goods, 3 yd. 2 ft. in length?

First Solution.

3 yd. 2 ft. = 11 ft.
1 ft. 3 in. = $1\frac{1}{4}$ ft. = $\frac{5}{4}$ ft.
$11 \div \frac{5}{4} = 11 \times \frac{4}{5} = \frac{44}{5} = 8\frac{4}{5}$ pieces.

Second Solution.

3 yd. 2 ft. = 11 ft. = 132 inches.
1 ft. 3 in. = 15 in.
$132 \div 15 = 8\frac{12}{15} = 8\frac{4}{5}$ pieces.

V. The Metric System

132. Present Use of Metric System in United States. The metric system today is more important in this country than ever before. Not only is it the system used in all scientific work, but it is also coming into more general use in industry because of the expansion of our foreign trade.

133. Teaching the Metric System in the Grades. Although the metric system has great practical importance to some people the chief justification for teaching it in the elementary schools is that some knowledge of it is needed by every educated person in order to be able to understand references that occur in newspapers and magazines. Wherever pos-

sible the elementary school should make its pupils familiar with the chief units of the metric system. Every school should be provided with such units and the pupils should become familiar with them through use. The pupils should have experience in making simple measurements using the metric units. Work of this kind is suggested in the exercise at the end of this chapter and in the chapter on mensuration. Little or no practice should be given on changing from the metric system into the English or vice versa as this is of no value to the ordinary person. The rough equivalents of the more important metric units in the English system should, however, be known. Thus the pupils should know that a meter is a little more than a yard or about 40 inches. Tables of various metric measures are given in the appendix for reference.

Exercise. Denominate Numbers

1. Reduce:
   1. 7 ft. 9 in. to inches.
   2. 7 yds. 2 ft. to feet.
   3. 87° 17′ 21″ to seconds.
   4. 8 lbs. 7 oz. to ounces.
   5. 5 gal. 2 qt. to quarts.
   6. 2 gal. 3 qt. 1 pt. to pints.
   7. 3 hrs. 10 min. 17 sec. to seconds.
   8. 8 mi. 238 ft. to feet.
   9. $\frac{5}{8}$ bu. to quarts.
   10. $\frac{5}{8}$ bu. to pecks and quarts.
   11. $\frac{2}{3}$ mi. to feet.
   12. $\frac{3}{5}$ gal. to quarts and pints.
   13. 32°.75 to degrees and minutes.
   14. $\frac{1}{2}$ yd. to feet and inches.
   15. $75\frac{1}{8}°$ to degrees, minutes and seconds.
   16. 347 ft. to yards and feet.
   17. 82 oz. to pounds and ounces.
   18. 71 pts. to gallons.

19. 372′ to degrees.
20. 18 ft. 9 in. to feet.
21. 5 yd. 2 ft. 8 in. to vards.
22. 8 lbs. 5 oz. to lbs.
23. 17 dys. 10 hrs. 15 minutes to hours.
24. 7 gal. 3 qt. 2 pt. to quarts.
25. 7568 sq. in. to sq. ft.
26. 49° 27′ to degrees.
27. 832 cu. ft. to cubic yards.
28. 8137 cu. in. to cubic ft.
29. 5876 seconds to hours, minutes and seconds.
30. 8 rd. 5 yd. 2 ft. to rods.
31. 5280 lbs. to tons and pounds.

2. Add:

1. 27° 32′ 18″
41° 50′ 46″

2. 2 bu. 1 pk. 5 qt.
3 bu. 3 pk. 4 qt.

3. 2 lb. 8 oz.
7 lb. 9 oz.
1 lb. 6 oz.

4. 9 hr. 7 min. 8 sec.
5 hr. 22 min. 17 sec.
1 hr. 48 min. 40 sec.

3. Subtract. Use Borrowing.

1. 8 yd. 1 ft. 7 in.
2 yd. 2 ft. 10 in.

2. 73° 18′ 20″
17° 27′ 32″

4. Subtract the examples under Exercise 3, using Equal Additions.

5. Multiply:

1. 8 ft. 7 in.
9

2. 3 lb. 9 oz.
4

6. Find the number of square feet of flooring in a room 19 ft. 7 in. by 12 ft. 8 in.

7. Divide.

1. 49° 35′ 17″÷2
2. 82 ft. 17 in.÷4
3. 7 lbs. 9 oz.÷6
4. 18 lb. 7 oz.÷2 lb. 2 oz.
5. 10 yd. 2 ft.÷1 yd. 2 ft.
6. 7 bu. 3 pk.÷2 pk.

CHAPTER X.

THE GEOMETRY OF FORM AND POSITION

I. Geometry in the Elementary School

134. General Statement. It is the duty of the elementary school to equip its pupils with a knowledge of those parts of elementary mathematics that have a directly practical, or general informational value. This means that the mathematics of the elementary school must be more than arithmetic, it must include the more practical and useful portions of geometry and algebra. The tendency at present is to make most or all of the mathematics of the high school elective. Where this is done it is necessary that the pupils get a little knowledge in the elementary schools of the portions of mathematics usually taught in the high schools in order that they may be able to decide intelligently whether or not they desire to study further in these subjects. A discussion of the algebra of the elementary schools will be found in Chapter XIII.

While the discussion of geometry and algebra given in this book is complete as far as it goes and covers as much of these subjects as can profitably be studied in the elementary schools, it does not by itself give sufficient preparation for the teaching of these subjects. No teacher should attempt to teach mathematics in the upper grades without first taking good courses in both algebra and geometry. The brief discussion given here is intended as a review for those who have already had such courses, and to suggest the methods of handling these subjects in the elementary school.

135. Three Kinds of Geometry. Applied geometry considers three questions concerning the objects studied, (1) Form, (2) Position, and (3) Size. Corresponding to these are three

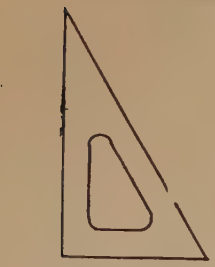
types of geometry, (1) The Geometry of Form, (2) The Geometry of Position, and (3) The Geometry of Size. The elementary phases of all three of these should be studied in the elementary school. The present chapter deals with the geometry of form and position, the following chapter deals with the geometry of size. Certain parts of the geometry of size and form have long been included in our arithmetic under the title of mensuration. The geometry of position, however, has usually been neglected entirely.

II. The Geometry of Form

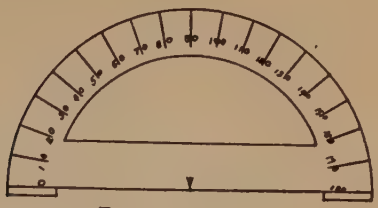
136. Historical and Present Uses of Geometrical Forms. Thousands of years ago the Babylonians, Egyptians and other early peoples made use of simple geometric forms and patterns in decorating walls and tile floors. Today the use of such forms is even more extensive and a knowledge of them is needed by everyone.

137. Preliminary Work. Starting with the first grade the pupils should start becoming familiar with the simple geometric forms that are most commonly used. They should learn to recognize these forms by noticing them as they occur in the school room and in the objects of their everyday observation and experience, and by using them in their construction work and in making decorations. No formal definitions should be given in the lower grades, the ability to recognize and use these simple forms being the object sought.

138. Simple Geometric Constructions. In the upper grades the simple geometric constructions should be taught and used in designing geometric patterns such as those given in the next section. The pupils should be provided with a ruler, a pair of compasses, a draftsman's triangle, and a protractor.



Draftsman's Triangle



Protractor

To Bisect a Line.

The middle point of a line can usually be found with sufficient accuracy by measuring. The following geometric construction is, however, sometimes useful.

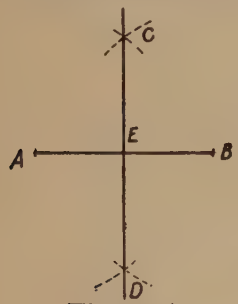


Figure 1.

With A as a center and any convenient radius greater than half of AB, draw arcs above and below AB as shown in the figure. Then with B as a center and the same radius draw arcs intersecting those already drawn at C and D. Draw CD.

The point E at which CD crosses AB is the mid-point of AB.

The line CD is perpendicular to AB at its mid-point and so is called the *perpendicular-bisector* of AB.

To Draw a Perpendicular to a Line From a Given Point.

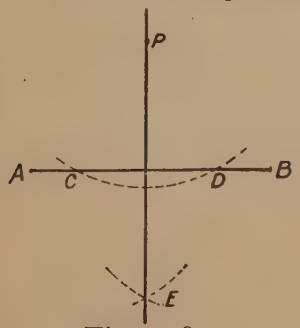


Figure 2.

Suppose it is required to draw a perpendicular to AB from P. With P as a center and any radius greater than the distance from P to the line AB, draw an arc intersecting AB at C and D.

With C and D as centers and any radius greater than half of CD draw arcs intersecting at E. Draw PE. PE is the required perpendicular.

To Draw a Perpendicular to a Line at a Point.

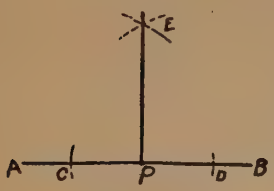


Figure 3.

Suppose it is desired to draw a perpendicular to AB at P. With P as center and any convenient radius draw arcs intersecting AB at C and D. With C and D as centers and any radius greater than half of CD draw arcs intersecting at E. Draw EP, which is the required perpendicular.

The practical method of drawing a perpendicular is to use a standard right angle and to copy or trace this whenever a perpendicular is required. The most convenient form of standard right angle is the draftsman's triangle previously mentioned. These may be obtained in various sizes. If they are not available a model right angle can be made by creasing a large sheet of heavy paper and then folding the crease back upon itself. With this right angle as a model the pupils can make a triangle of heavy cardboard.

Perpendiculars can also be drawn by means of the protractor. Methods of laying off perpendiculars in field work are given in Sections 171 and 189.

To Bisect an Angle.

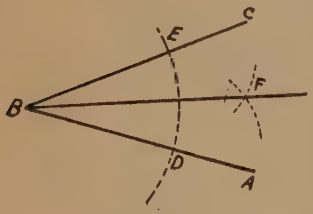


Figure 4.

To bisect the angle ABC, with B as a center and any convenient radius draw an arc cutting the sides of the angle at D and E. With D and E as centers and a radius greater than half of DE draw arcs intersecting at F. Draw FB, which is the bisector of the angle ABC.

An angle may often be bisected with sufficient accuracy by means of a protractor.

To Construct an Angle Equal to a Given Angle.

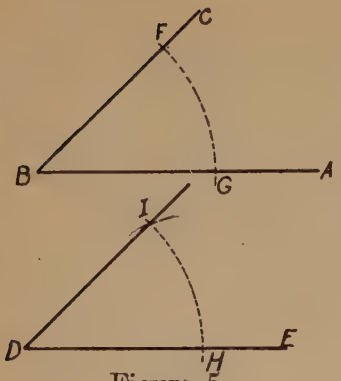


Figure 5.

Suppose it is desired to draw another angle equal to angle ABC. Draw any line as DE. Then with B as a center and any convenient radius draw an arc cutting the sides of the angle at F and G. With D as a center and the same radius draw an arc as shown in the figure cutting DE at H. With H as a center and a radius equal to the distance FG draw an arc intersecting the last arc drawn at I. Draw ID. The angle IDE is equal to the angle ABC.

In practical work an angle can often be drawn equal to a given angle with sufficient accuracy by means of a protractor.

To Draw a Line Parallel to a Given Line, and Passing Through a Given Point.

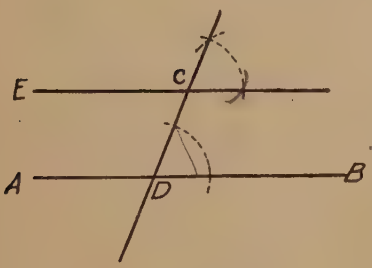


Figure 6.

Suppose it is required to draw a line through C and parallel to AB. Draw any line through C cutting AB at D. At C construct an angle equal to the angle CDB. The line CE is the required parallel.

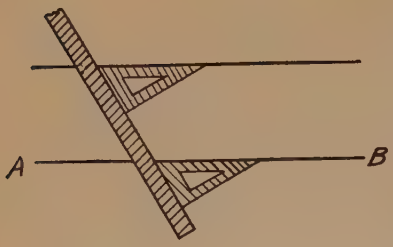


Figure 7.

In practical work parallel lines are often drawn by means of a draftsman's triangle as shown in the accompanying figure. Suppose it is required to draw a line parallel to AB. Place the triangle so that its hypotenuse coincides with the line AB. Place a ruler along one of the other sides of the triangle. Holding the ruler firmly slide the triangle along the ruler. The hypotenuse of the triangle is always parallel to the line AB.

To Construct a Triangle When the Sides are Known.

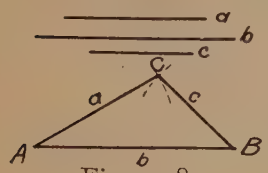


Figure 8.

Suppose the three sides of the triangle are to be equal to *a*, *b*, and *c* respectively. Draw a line AB equal to *b*, with A as a center and *a* as a radius, draw an arc as shown in the figure. With B as the center and *c* as a radius draw an arc intersecting the first arc drawn at C. Draw AC and BC. ABC is the required triangle.

To Divide a Line into Any Number of Equal Parts.

If the division of a line into a number of equal parts can not be made with sufficient accuracy by measuring the following construction can be used.

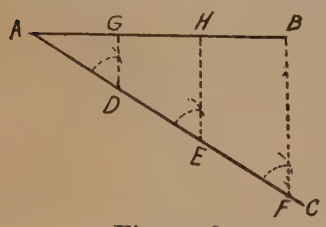


Figure 9.

Suppose it is desired to divide AB into three equal parts. Draw a line AC from A as shown in figure. Mark off three equal distances on AC as AD, DE and DF. Draw FB. Draw EH and DG parallel to FB. The points G and H divide AB into three equal parts.

Another method that can sometimes be used is as follows.

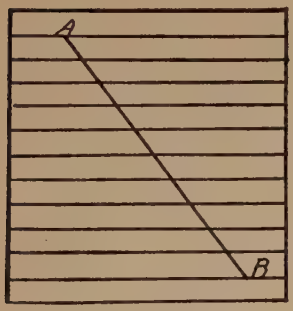


Figure 10.

First rule a large sheet of paper with parallel lines at equal distances apart. If a line AB is to be divided into ten equal parts, set the compasses so that the distance between the points equals AB. Place one point of the compass on the first line, at A in figure and turn the compasses until the other point just touches the tenth line below, at B in the figure. Draw AB. The parallel lines divide AB into ten equal parts.

To Divide a Circumference Into Four Equal Parts.

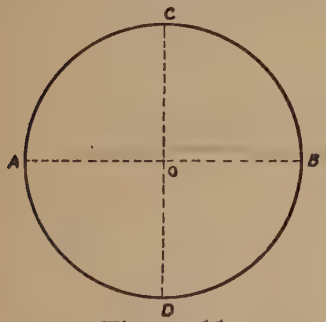


Figure 11.

Draw the diameter AB. Construct CD perpendicular to AB at O the center of the circle. The points A, C, B and D divide the circumference into four equal parts.

To Divide a Circumference Into Six Equal Parts.

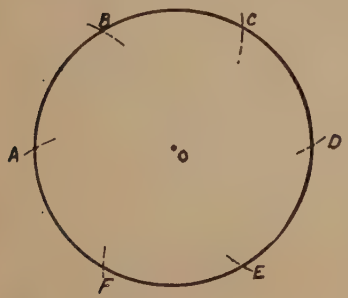
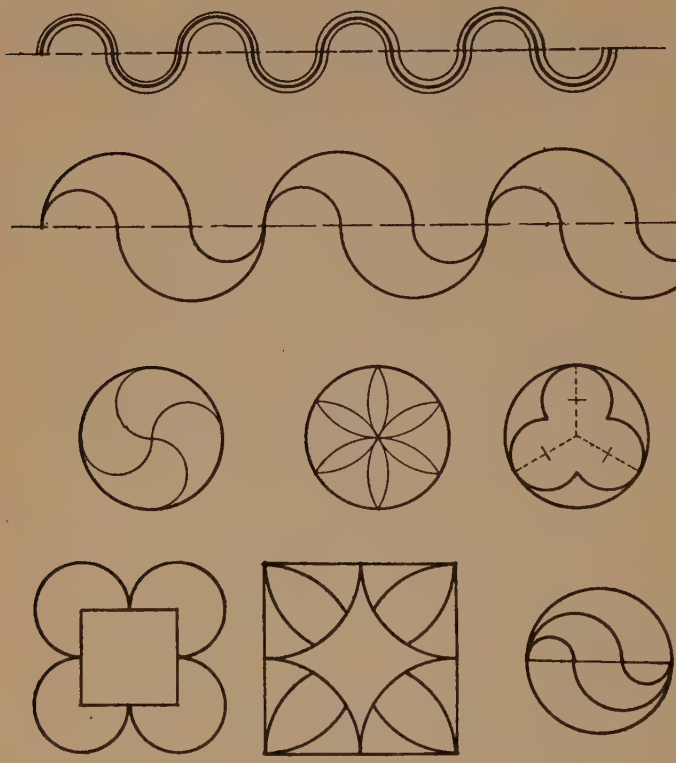


Figure 12.

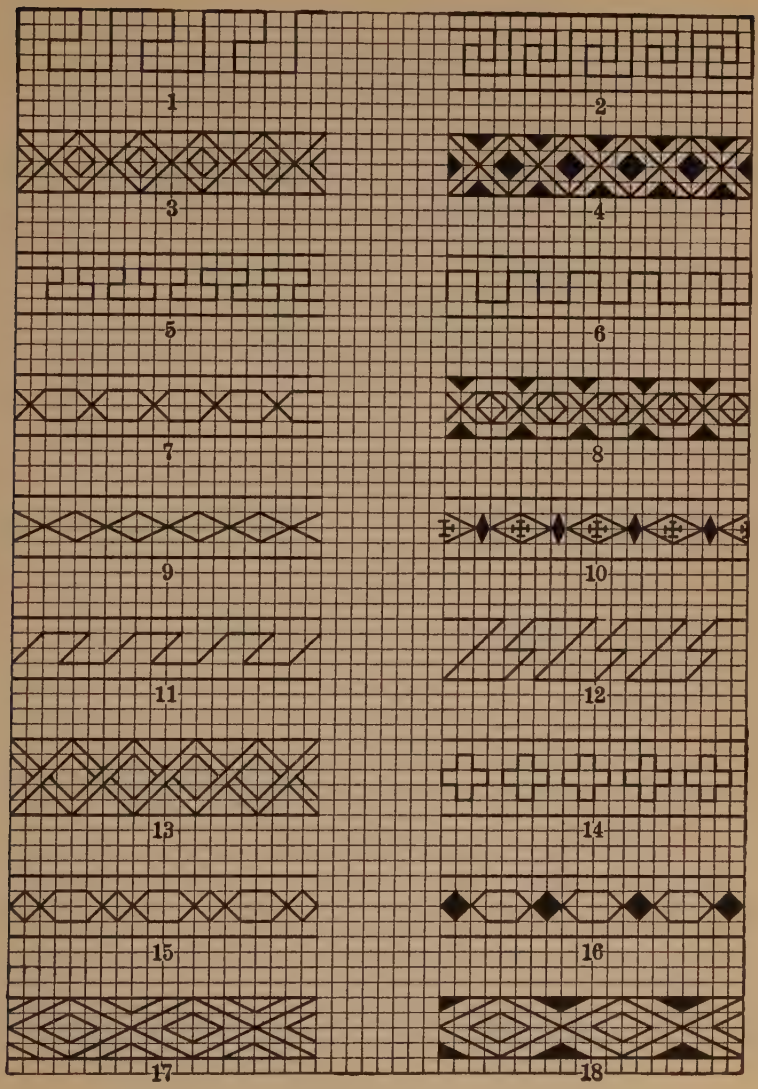
Start with any point A on the circle as a center and with a radius equal to AO, the radius of the circle, draw an arc cutting the circle at B. With B as center and the same radius draw an arc cutting the circle at C, etc. This will divide the circle into six equal parts.

139. Geometric Patterns. The following geometric patterns and designs can all be made by use of the various constructions given in the preceding section.



Many beautiful patterns can be made with the aid of paper ruled in squares, known as cross section paper. The designs given on the next two pages are reprinted with permission from Plane and Solid Geometry, by J. H. and K. P. Williams.

140. Ratio and Proportion. In practical work it is often necessary to compare one number or quantity to another. Numbers may be compared by means of two different arithmetical processes, subtraction and division. By subtraction we can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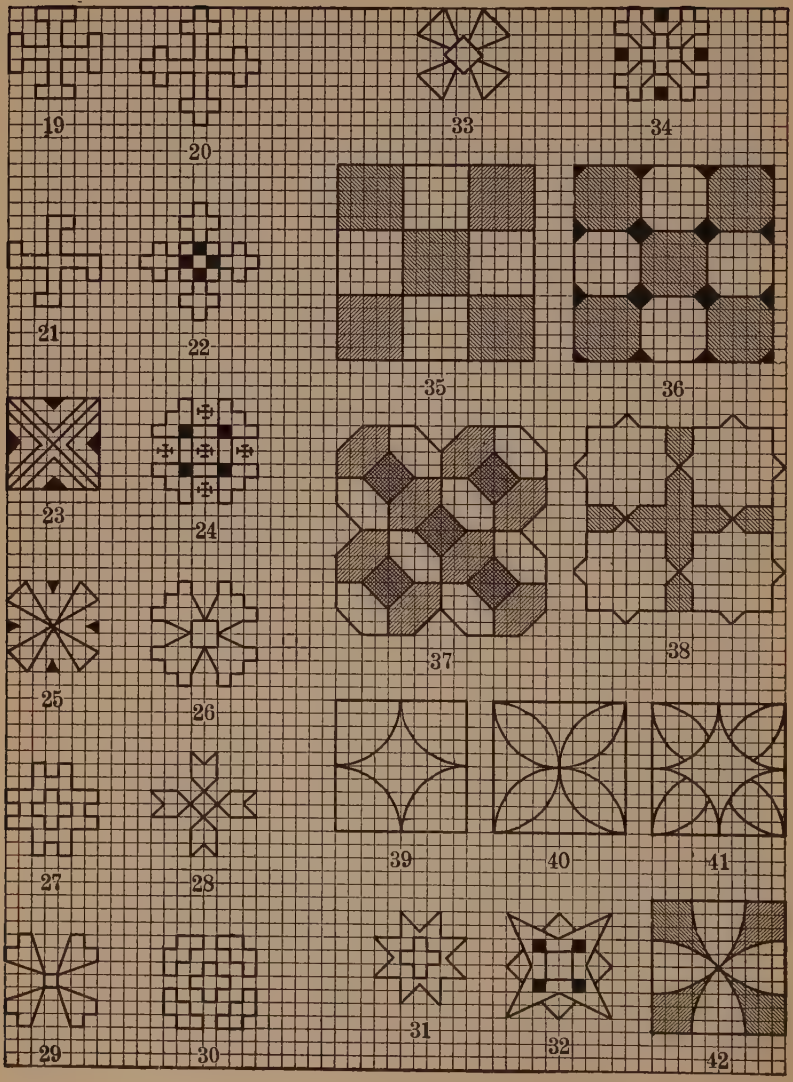
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find the difference between the two numbers, that is how much larger or smaller one is than the other. Thus 12—9=3, tells us that 12 is 3 more than 9. By division we can find the *ratio* of the two numbers, that is we can find that the one number is so many times as large as, or a certain fractional part of the other. Thus 12÷4=3 tells us that 12 is just 3 times as large as 4, and 8÷12=$\frac{8}{12}$=$\frac{2}{3}$ tells us that 8 is just two thirds of 12. The most convenient way to write a ratio is as a common fraction. Thus the ratio of 9 to 24 is written $\frac{9}{24}$, or $\frac{3}{8}$.

The pupils in the elementary school should have plenty of practice in comparing numbers both by subtraction and division from the first or second grade, and the terms difference and ratio should be thoroughly familiar before the upper grades are reached.

If four quantities are so connected that the ratio of the first to the second is the same as the ratio of the third to the fourth then the four quantities form a *proportion.* A proportion is, therefore, two equal ratios or two equal fractions. The fact that the ratio of a to b is the same as the ratio of c to d is written $\frac{a}{b}=\frac{c}{d}$ and is read either "a is to b as c is to d," or "a over b equals c over d."

In practical applications of proportion three of the quantities involved are usually known and the fourth is to be found. In elementary work it is easiest to write the unknown quantity as the first term of the proportion or as the numerator of the first fraction.

No rules are necessary for the formation or solution of proportions. The pupils should form the proportion by thinking of the relations existing between the quantities, and writing down the two equal relations or ratios. In solving the proportion $\frac{AB}{3}=\frac{8}{7}$ the pupils should see that if $\frac{1}{3}$ of AB equals $\frac{8}{7}$, then AB=3×$\frac{8}{7}$=$2\frac{4}{7}$=$3\frac{3}{7}$, or if the pupils understand the

solution of simple algebraic equations, AB in the above proportion can be found by multiplying both sides of the equation by 3.

The old forms of writing ratios and proportions, as 3:8 for $\frac{3}{8}$, and $a:b::c:d$ for $\frac{a}{b}=\frac{c}{d}$, are clumsy and difficult to work with and are rapidly dropping out of use. They should not be used at all in the elementary schools.

141. Similarity of Shape. Geometric figures may be of different sizes and still have the same *shape*. Figures that have the same shape are said to be *similar*. It is evident that any two circles are similar and any two squares, as they have the same shape. Two rectangles, however, may or may not be similar. Thus, in Figure 13, rectangles 1 and 2 evidently do not have the same shape but rectangles 2 and 3 do and are therefore similar.

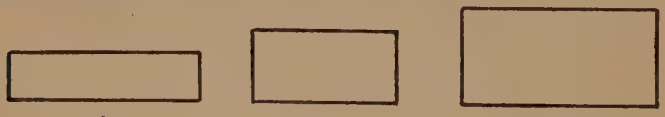


Figure 13.

The shape of a polygon of a given number of sides depends on two things, the size of the angles and the relative length of the sides. Any change in the sizes of the angles changes the shape of the figure. In Figure 14, the two parallelograms

Figure 14.

have equal sides but unequal angles and are not the same shape.

If the sides of a figure are all made longer in the same proportion and the angles are left unchanged the figure will be

made larger but the shape will not be changed. In Figure 15, the sides of the second triangle are each just twice as long as the corresponding sides of the first, but the angles of the two triangles are equal. It is seen that the two triangles have the same shape or are similar.

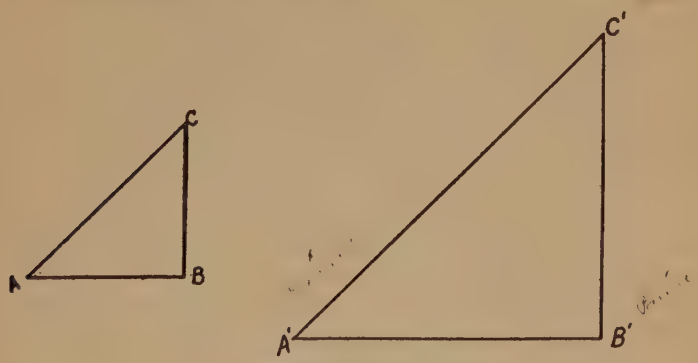


Figure 15.

From the above discussion it is seen that: *In similar polygons the corresponding angles are equal and the corresponding sides proportional.*

Thus, in Figure 15, the two triangles are similar and angle A=angle A′, angle B=angle B′, angle C=angle C′, and $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$.

142. Drawing to Scale. One of the most important applications of similar figures is to the making of scale drawings. Such drawings are made the same shape as the original but smaller. Thus in drawing a plan of the school room, if it is a rectangle 30 ft. by 40 ft., we would draw a similar figure using some convenient *scale,* perhaps letting 1 inch on the drawing represent 10 feet. Our scale drawing would then be

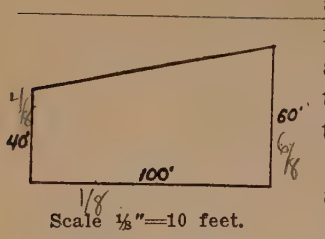


Figure 16.

a rectangle 3 in. by 4 in. Figure 16 is a scale drawing of a city lot. The angles of the figure are the same as the angles of the lot and ⅛ inch on the figure represents 10 feet.

Since the angles of the drawing are made equal to the angles of the original figure and the sides are made proportional by using the same scale throughout the drawing, such a drawing has the same shape and proportions as the original.

Exercise. Scale Drawings

1. If a figure is drawn to the scale of 1 inch=1 foot, what length on the original does a line on the drawing 4½ inches long represent?

2. If the scale is 1 inch=5 feet, a line 1¾ inches long on the drawing represents what length on the original?

3. Find the dimensions of the lot shown in Figure 17. Also find the dimensions of the house, of the garage and the garden.

4. If a room is 15′x12′ what will be the corresponding dimensions on a scale drawing, scale 1 inch=6 feet?

5. Measure the following and make scale drawings of each. Use any suitable scale.

1. The top of your desk.
2. The school room.
3. The school building.
4. The school yard.

III. The Geometry of Position

143. General Statement. In the accompanying figure is given a scale drawing of a city lot showing the location of the house, garage, walk, drive, garden and trees. In order to locate these accurately on the drawing it is necessary in some way to locate their position on the lot. The following methods of locating the position of an object or a point on a plane will be found useful in this connection.

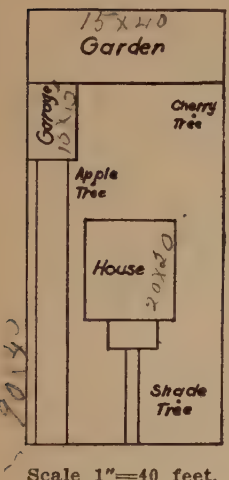


Figure 17.

144. Locating a Point by Measuring its Distance from Each of Two Lines. One of the commonest ways of locating a point is by measuring its distance from each of two lines. Thus the position of the cherry tree in Figure 17 might be determined by measuring its distance from the back and from the nearest side of the lot.

If a point is known to be 6 inches from line AB in Figure 18 and 4 inches from CD it can be located as follows. All points 6 inches from AB evidently lie on a line parallel to AB and 6 inches from it. There are two such lines, on either side of AB. All points 4 inches from CD will lie on lines parallel to CD and 4 inches from it on either side. So, to located a point 6 inches from AB and 4 inches from CD (See Figure 18), draw two lines parallel to AB and 6 inches from it on either side and two lines parallel to CD and 4 inches from it on either side. The points E, F, G, and H, where these lines intersect are evidently 6 inches from AB and 4 inches from CD. In practice it is usually known on which side of the lines the point lies so it is usually sufficient to draw only the parallels on the proper sides of the given lines. In this way only one point is located, instead of four.

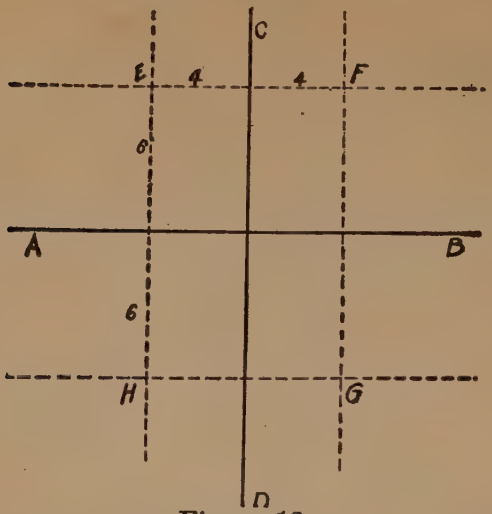


Figure 18.

145. Locating a Point by Measuring its Distance from each of Two Other Points. A point may be located by measuring its distance from each of two points whose positions are known. Thus, the cherry tree in Figure 17 could be located by measuring its distance from each of the rear corners of the lot. If a point is known to be 7 inches from A and 10 inches from B in Figure 19, it can be located as follows. All points 7 inches from A evidently lie on a circle having A as a center and a radius of 7 inches. All points 10 inches from B lie on a circle having B as a center and a radius of 10 inches. The two points C and D, where these two circles intersect are evidently each 7 inches from A and 10 inches from B. Which of these is the desired point is usually evident in practice from the conditions of the problem.

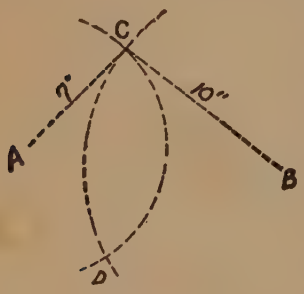


Figure 19.

146. Locating a Point by Measuring its Distance from a Known Line and from a Known Point. A point may also be located by measuring its distance from a known line and from a known point. If a point is known to be 6 inches from line AB (Figure 20) and 10 inches from C it may be located as follows. Draw lines parallel to AB and 6 inches from it on either side. Draw a circle with C as a center and a radius of 10 inches. The points where the line and the circle intersect (D and E in the figure) are evidently 6 inches from AB and 10 inches from C. In general four points are determined by this construction. In practice it is usually possible to tell from the conditions of the problem which of these is the desired point.

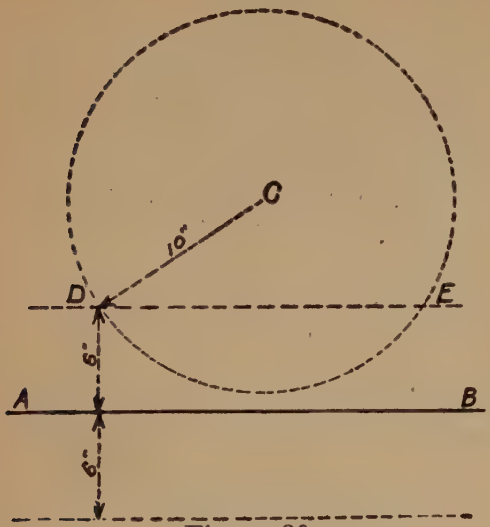


Figure 20.

147. Locating a Point by Measuring its Distance and Direction From a Given Point. A point may also be located by measuring its distance from a given point and its direction as referred to a given line. In Figure 21 the position of a point such as P might be determined by measuring the distance AP and the angle PAB.

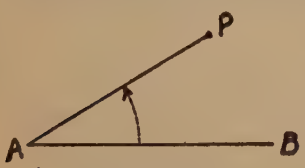


Figure 21.

Figure 22.

The north and south line through a point can be determined by means of a pocket compass and the direction of any other line can be determined by measuring the angle it makes with the north and south line. Thus the point P can be located in Figure 22 by measuring the line AP and the angle PAN which AP makes with the north and south line NS through the point A.

148. Points Equidistant from Two Lines. It is sometimes necessary to locate points that are equally distant from two lines. If the lines are parallel it is evident that all points at the same distance from each lie on the line parallel to them and half way between. If the lines intersect, all points at the same distance from each evidently lie on the bisector of the angle between them. The method of locating the bisector of an angle is given in Section 138.

149. Points Equidistant from Two Points. It is sometimes necessary to locate points that are at the same distance from each of two given points. It is evident that one point equally distant from A and B, in Figure 23, is C the mid-point of the line joining A and B. Further any point immediately above or below C is also the same distance from A and B. It is evident, therefore, that all points equally distant from A

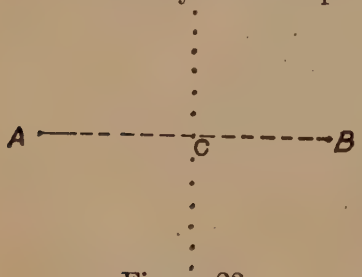


Figure 23.

and B lie on a line perpendicular to the line joining A and B at its mid-point, that is on the perpendicular bisector of AB. The method of determining the perpendicular bisector of a line was given in Section 138.

EXERCISE. PLANS AND MAP DRAWING. LOCATION OF POINTS.

1. Make a plan of your school room, to scale, showing the location of the teacher's desk, the students' desks, etc.

2. Make a map of your school yard showing the location of the various buildings, trees, etc.

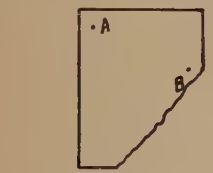


Figure 24.

3. Figure 24 is a scale drawing of a county. Find the distance between the two towns A and B. The county seat C is 15 miles from A and 10 miles from B. Make a copy of the map on a larger scale and locate the county seat.

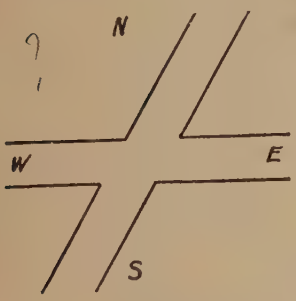


Figure 25.

4. Two streets intersect as shown in Figure 25. Each street is 30 ft. wide. A water main runs under the east and west street and is known to be located 10 feet from the south curb of the street. A branch main runs under the center of the other street. Workingmen wish to locate the junction of the two mains in order to make repairs. Draw a figure to scale and locate the point where the workingmen should dig.

5. On your enlarged copy of the county map (Exercise 3) locate town D which is 10 miles from A and 8 miles from the northern boundary of the county.

6. On the same map locate village E, which is 5 miles south-west of C.

7. If possible procure an outline map of your state and locate the principal cities by taking the proper measurements on a complete map of the state.

8. In a factory it is desired to place a light so as to illuminate two machines that are 5 feet apart. Make a scale drawing showing where the light should be placed if the type of light used gives the desired degree of illumination when placed 8 feet from the machine in a horizontal direction.

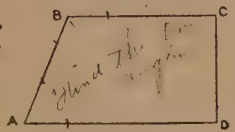
9. 

Figure 26.

The lawn by the side of a public building is the shape shown in the figure. A circular flower bed is to be placed so as to be equally distant from the three streets AB, BC and AD. Make a copy of the drawing and locate the center of the flower bed.

CHAPTER XI.

THE GEOMETRY OF SIZE.

I. The Theory and History of Measurement.

150. The History and Meaning of Measurement. The need for measuring things goes back to very early times. As long as useful commodities were so plentiful that each individual could have all that he wanted for the taking there was no necessity for counting or for measuring. As soon, however, as certain commodities became scarcer so that all could not have as much as they wanted private ownership was introduced and counting and measuring became necessary in order to be able to separate one individual's belongings from those of another.

Geometry probably originated with the Egyptians and grew out of the necessity of measuring land. The annual overflow of the Nile washed out all boundaries so that it was necessary each year to measure the land in order to again fix the boundaries between one man's property and that of his neighbors. As early as 1400 B. C. Egypt was divided into equal rectangular plots of ground.

In order to measure any quantity it must be compared to some other quantity of the same kind which is known as the *unit of measure.* At first the world used rough units of measure which were usually derived from parts of the human body or from some other natural source. Thus almost every civilized nation has had in the past a unit of length called the foot. Undoubtedly this unit was originally derived from the length of the human foot. At first the foot of the person taking the measurement was used as the unit of measure, but as the commodities measured became more valuable and more accurate measurements were needed legislation was passed defining the

length of the foot. These definitions were at first local so that the foot of one country or district was not the same as that of another. Today the various units of measure in common use in this and other countries have all been accurately defined and mean the same the world over. Thus the yard, the standard unit of length in the English system, is defined as the distance at 62°F. between the center marks in two gold plugs set in a large bar kept in the Standards Department of the British Board of Trade. A duplicate of this bar is kept in the United States. The meter, the standard unit of length in the metric system, is defined as the distance at 0°C. between two lines ruled on a bar of platinum that is kept at the International Bureau of Weights and Measures near Paris.

A unit of measure that has been accurately defined and that is widely known and used is called a *standard unit of measure.*

The process of measuring any quantity is always one of comparing the quantity to be measured to standard units of the same kind. Thus to measure a length we compare it to some length that has been adopted as a standard, and to measure a weight we compare it to some standard weight. To measure the length of a table we compare it to some standard length, say the length of a foot rule, and if the table is four times as long as the rule we say its length is 4 feet.

151. Accuracy of Measurements. No measurement can ever be absolutely accurate. We can measure the length of this page to the nearest inch, to the nearest eighth of an inch, to the nearest thirty-second, or even, with the proper instruments and care, to the nearest hundredth of an inch, but no matter how refined the methods and instruments used, we can never measure its length exactly but only to a certain degree of accuracy. Every number that is obtained as the result of a measurement is therefore only an approximate number and it is very important to know how closely it approximates absolute accuracy or to what degree of accuracy the measurement

has been taken. Thus if we say a line is 10 feet long it may mean that it has been measured roughly and found to be nearer to 10 feet long than to 9 feet, or 11 ft., or it may mean that it has been measured more accurately, say to the nearest inch, and is nearer to 10 feet than to 9 feet 11 in. or to 10 feet 1 in. It is very important in stating the result of a measurement to state the degree of accuracy with which the measurement has been taken.

152. Accuracy of Results. In working with numbers that are the results of measurements, that is with approximate numbers, it is very important to remember that the results of any mathematical calculation can be no more accurate than the numbers used in the calculations. If the length and width of a rectangular plot of ground are measured and found to be 150.7 ft. and 49.2 ft. respectively, to the nearest tenth of a foot and the area is computed, it is found to be 7414.44 sq. ft. It should not be supposed however that the area has been determined to the nearest hundredth of a square foot and to use the entire result obtained is to deceive oneself and others. As a matter of fact, the last three figures of the above result are absolutely unreliable and all that we really know about the area is that it is probably about 7410 sq. ft. and probably varies from that amount by less than 10 sq. ft. The truth of this may be seen from the following. The length of the field is not exactly 150.7 ft., but only that to the nearest tenth of a foot. In fact it may actually be anything from 150.65 ft. to 150.74 ft. inclusive. The width is not exactly 49.2 ft. but may be anything from 49.15 ft. to 49.24 ft. inclusive. The area may, therefore, be anything from $49.15 \times 150.65 = 7404.4475$ sq. ft., to $41.24 \times 150.74 = 7422.4376$ sq. ft. By comparing these two results it is seen that the first two figures of the number representing the area are undoubtedly a 7 and a 4 respectively, the third figure may be anything from 0 to 2 inclusive with 1 as the most probable figure. The remaining figures are entirely unreliable, so all we are justified in saying about the area

is that it is between 7400 sq. ft. and 7420 sq. ft. with 7410 sq. ft. as the most probable figure, as it probably differs from the exact area by less than 10 sq. ft.

In the above illustration the width of the field was expressed by a number of three figures and the length by a number of four figures and it was found that only the first three figures of the result were at all reliable. This gives us a rough working rule which enables us to tell roughly how many figures of the result are reliable.

In calculating with approximate numbers only as many figures of the result should be saved as there are figures in the least accurate number (the one having fewest figures) entering into the calculations.

Thus if 37.89 and 573.27 are two approximate numbers and any arithmetical calculations are performed with these numbers, only the first four figures of the result will be reliable as 37.89 has only four figures.

Not only is it important to realize that calculations based upon approximate data will give only approximate results, but it is also worth while to remember that where only an approximate result is wanted much time and labor can be saved by using only approximate data. Thus if the product of 3675.3 and 9966.779 is desired, approximate to three places only, it is not necessary to use all of the figures given in the data but it will be sufficiently accurate to use only four figures, using the nearest figure in the fourth place, and finding the product of 3675 and 9967.

In general, where the data is more accurate than the result desired, use only one more place in the data than desired in the result, using the nearest figure in the last place of the data that is saved.

153. Measurement in the Elementary Schools. The work in mensuration in the elementary schools should be based almost entirely on measurements made by the pupils themselves. There is little practical value in finding the area of a

triangle having a base of 17 inches and an altitude of 9 inches if these dimensions are given by the book or teacher, and ability to find the area when the dimensions are given does not necessarily imply or prove the ability to find the area of a figure where the proper dimensions must be obtained by measurement. Pupils should start making simple measurements, using standard units, as early as the first grade and this work should be continued throughout the grades and serve as the basis for the work in mensuration.

The formulas or rules of mensuration should be discovered by the pupils for themselves but once discovered they must be memorized and applied to actual figures.

154. Use of Models. In discovering the rules for finding the areas and volumes of the simple geometric figures, models should be used. Models of wood may be bought at a reasonable figure, or cardboard models may be made by the teacher and pupils.

EXERCISE. MEASUREMENT.

1. Why must the unit of measure always be the same kind of quantity as the quantity being measured?

2. Measure the following:
   1. The length of this page to the nearest sixteenth of an inch.
   2. The length and width of some room to the nearest inch.
   3. The length of this page to the nearest millimeter.
   4. The length and width of a room to the nearest centimeter.

3. If 3.8 and 12.2 are approximate numbers show that not more than two figures in the quotient 3.8÷12.2 are at all reliable.

Note: 3.8 may be anything from 3.75 to 3.84 and 12.2 anything from 12.15 to 12.24. Therefore the smallest possible quotient is 3.75÷12.24 and the largest is 3.84÷12.15.

4. If 59.75 and 28.27 are approximate numbers how many figures in their sum are reliable? In their difference? Prove each statement.

5. Multiply 9966.779 by 3675.3. Multiply 9967 by 3675. How do the first three figures of the two results compare?

6. Divide 5,788,738 by 83,957. Divide 5,789,000 by 83,960. How do the first three places of the two results compare? What general conclusion do you draw from Exercises 5 and 6?

II. Areas.

155. The Measurement of Areas. In order to measure any area we must have a unit of measure of the same kind, namely an area. The area that is usually used as a unit is the area of a square whose side is one linear unit. Common units of area are the square inch, the square foot, etc. The *square inch* is the *area* of a square one inch on a side. The *square foot* is the *area* of a square one foot on a side, etc.

156. The Area of the Rectangle. The rectangle is the only figure the area of which can be obtained by direct measurement, that is by applying the unit of area and counting the number of times it is contained in the area measured. The practical methods of finding the areas of all other figures are based directly or indirectly on the measurement of the rectangle.

As early as the third grade the pupils should have practice in finding the areas of rectangles, whose dimensions are each an even number of inches, by covering with square inches cut out of paper, or by folding into square inches. In either case the area is determined by counting the number of square inches in the rectangle.

In the fifth grade the pupils should review the previous work and from it discover the short method of finding the area of a rectangle namely by multiplying the length by the width. This discovery may be made in either of two ways. The terms *base* and *altitude* should be introduced at this time.

Method 1. Each pupil is supplied with several rectangular pieces of cardboard whose dimensions are all an integral number of inches, with a supply of square inches cut from cardboard, and with a foot rule. The teacher reviews the meaning of measuring and sees that the pupils know that to measure is to find how many times some standard unit is contained in the magnitude to be measured. The pupils already have had some experience in finding areas of rectangles by folding into square inches, or by fitting on square inches and counting, and this is reviewed at this time. The pupils take rectangles 3 inches by 4 inches, and find the area by covering with square inches and counting. Their rectangles then look as follows:

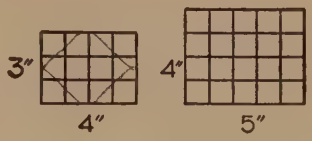
Next the teacher points out that this method of finding the area is long and inconvenient, and interests the pupils in finding a shorter method. The pupils already know how to find the number of desks in the room, or the number of postage stamps in a sheet without counting them one by one. This knowledge is reviewed and applied to the problem in hand, leading the pupils to see that "The number of square inches in the rectangle equals the number of rows times the number of square inches in one row."

Next, the possibility of finding how many rows there will be and how many square inches in each row without fitting on the square inches is considered and the pupils are led to see that, since each square inch is one inch wide, there will always be as many square inches in one row as there are inches in the base; and, since each row is one inch high there will always be as many rows as there are inches in the altitude.

From the above the pupils form the hypothesis that the area of a rectangle may be found by multiplying the number of inches in the base by the number of inches in the altitude. This hypothesis is verified by taking the other rectangles, measuring the base and altitude of each with the foot rule and multiplying. The result thus obtained is then checked by fitting on square inches and counting. Having verified their hypothesis the pupils formulate the rule.

The area of a rectangle is equal to the product of the base and the altitude, or $A=b\times a$.

Method 2. The pupils are supplied with the same material as in the previous method. They first find the areas of two or more rectangles by covering them with square inches and counting. Each pupil then has before him two or more rectangles as follows:



Again the question of finding an easy method for determining the area is raised. In order to do this the teacher calls the pupils' attention to the altitude, base and area of each rectangle and they see that the altitude of the first is 3 inches, its base 4 inches and its area 12 inches. In the second rectangle they see that the altitude is 4 inches, the base 5 inchs and the area 20 square inches. By noting the relation between the altitude, base, and area in the two rectangles, the pupils form the same hypothesis as in the previous method and verify it in the same way.

The square should next be considered as a special case of the rectangle. If we call the side of a square s we have a rectangle with base equal to s and altitude equal to s, so the area is $s\times s$ or s^2.

The area of a square is equal to the square of its side, or $A=s^2$.

157. The Area of the Parallelogram. The rule for the area of the parallelogram can not be obtained by direct measurement, but is derived from the rule for the area of the rectangle. The pupils should cut parallelograms out of paper similar to that shown in the accompanying figure ABCD. By folding back the part BCE so that CE is folded back on ED and the crease goes through B, the line BE is determined. Then the triangle BEC can be cut off and put on the other end of the figure, changing the parallelogram into a rectangle. The area of a rectangle is equal to the product of its base and altitude, so the area of the rectangle formed from the parallelogram is AB×BE (Figure 28). The area of the parallelogram is the same as that of the rectangle as nothing was added or taken away so the area of the parallelogram also equals AB×BE. But AB is the *base* of the parallelogram and EB is the *altitude* (Figure 27). Therefore:

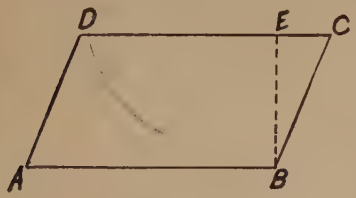


Figure 27.

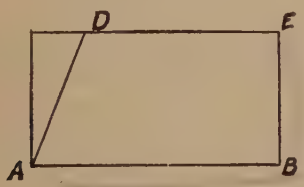


Figure 28.

The area of a parallelogram is equal to the product of its base and its altitude, or $A=b\times a$.

The letters A, B, C, D, and E are used here for convenience. The pupils instead of talking about the lines AB and BE may simply say *this line* and *this line,* indicating the line referred to by tracing it from end to end with pencil or pointer.

158. Drawing the Altitude. In order to be able to find areas the pupils must have some way of locating the altitude

of a figure. Various methods of drawing perpendiculars were described in Section 138.

159. The Area of the Triangle. The rule for finding the area of a triangle may be obtained directly from the rule for the area of the rectangle, but it is easier to obtain it from the rule for the area of the parallelogram. The pupils should be provided with cardboard or paper triangles similar to ABC in Figure 29. They can then make another triangle exactly equal to ABC by tracing on paper and cutting it out. This new triangle may be placed with ABC so as to form a parallelogram as shown in Figure 30. The area of a parallelogram is equal to the product of its base and its altitude, so the altitude CD of the parallelogram is drawn, then the area of the parallelogram is AB×DC (Figure 30). The area of the triangle is half that of the parallelogram, as the latter is composed of two equal triangles so the area of the triangle is ½×AB×DC. But AB is the base of the triangle and DC is the altitude (Figure 30). Therefore:

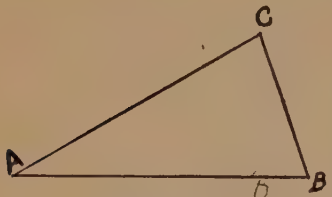


Figure 29.

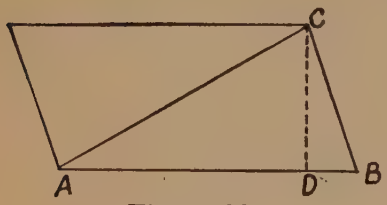


Figure 30.

The area of a triangle is equal to one half the product of the base by the altitude, or $A=\frac{1}{2}b\times a$.

160. The Area of the Trapezoid. The rule for the area of the trapezoid may be obtained from the rectangle, the paral-

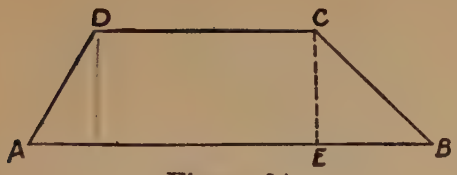


Figure 31.

lelogram, or the triangle. Probably the easiest method is to obtain it from the rule for the area of the parallelogram. The pupils should be provided with models similar to Figure 31. DC is called the upper base, AB the lower base and CE the altitude. By tracing the pupils make another trapezoid equal to ABCD, and the two can be placed together so as to form a parallelogram as in Figure 32. The area of a parallelo-

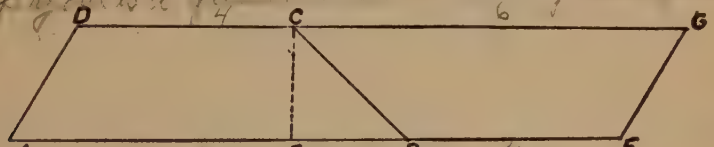


Figure 32.

gram is equal to the product of its base and altitude so the area of the parallelogram in Figure 32 is equal to AF×CE. The trapezoid is just one-half of the parallelogram as the parallelogram is made up of two equal trapezoids, so the area of the trapezoid is equal to ½×AF×CE (Figure 32). But AF=AB+BF=AB+DC, or AF is the sum of the two bases of the trapezoid, and CE is the altitude of the trapezoid (Figure 31). Therefore:

The area of a trapezoid is equal to half the product of its altitude and the sum of its bases, or $A=\frac{1}{2}\ a\ (b+b')$.

161. The Areas of Polygons. The areas of polygons other than rectangles, parallelograms, triangles and trapezoids may be found by first dividing them into triangles, rectangles,

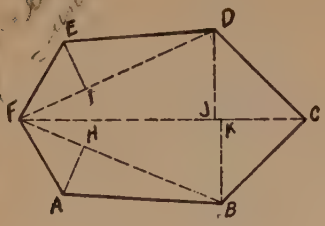


Figure 33.

and trapezoids. Two methods are described below.

Method 1. From any vertex as F (Figure 33) draw all possible diagonals thus dividing the polygon into triangles. Draw the altitude of each of the triangles and measure the altitudes and bases. Find area of each of the triangles and add.

Method 2. Draw the longest diagonal as FC (Figure 34) and then draw perpendiculars to it from each of the other vertices as EG, AH, DI, and BJ. This divides the figure into right triangles and trapezoids. GIDE is a trapezoid because EG and DI are parallel, each being perpendicular to FC. The two parallel sides EG and DI are the bases and GI is the altitude, being the perpendicular distance between the bases. In the same way ABJH is a trapezoid with AH and BJ as bases and HJ as altitude. If the perpendiculars EG, DI, AH and BJ and the parts of the diagonal FG, GH, HJ, JI, and IC are measured the areas of all of the triangles and trapezoids may be found.

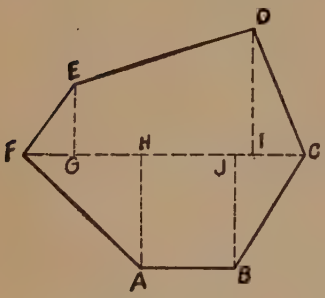


Figure 34.

Exercises and Problems. Areas of Polygons.

1. Find the area of your school room.

2. Find the number of acres in the school grounds or in some convenient tract of land.

3. Define altitude of a triangle. How many altitudes does a triangle have?

4. Construct a triangle with sides 3″, 3.5″ and 5.5″, respectively. Draw all of the altitudes using a draftsman's triangle or a model right angle.

5. Measure an altitude and the corresponding base of the triangle in Problem 4 to the nearest tenth of an inch and find the area. How many figures of this result are reliable? As a check, measure another altitude and base and again find the area. If your two results differ by more than 1 in the last reliable figure, repeat your measurements and calculations.

6. Draw each of the following on a large sheet of paper. Find the area in two ways. If your two results differ by more than 1 in the last reliable figure, repeat your measurements and calculations.

a. An obtuse angled triangle.

b. A trapezoid. Find the area by means of the formula $A=\frac{1}{2}\,a(b+b')$. As a check divide the trapezoid into two triangles by drawing a diagonal. Find the area of the triangles and add.

c. An irregular polygon of four sides. Divide into two triangles by drawing a diagonal. Measure and compute area. As a check draw the other diagonal dividing the figure into two other triangles. Measure and compute area.

d. An irregular polygon of five sides. First find the area by the first method of Section 161, and then as a check find the area by second method of that section.

e. An irregular polygon of seven sides. Proceed as in (d).

7. If possible, find the area of some irregularly shaped field or plot of ground in the neighborhood of the school.

8. Measure your school room and determine the cost of:

a. Covering the floor with linoleum.

b. Papering the walls.

Get the prices and other necessary figures from local merchants and workmen.

9. A certain make of tin shingles are 9″ by 14″. When laid they are exposed to the weather 8″ by 12″. How much will it

cost to cover a roof 26′ by 34′, at $8 a square (100 sq ft.)? As a check, work in two ways.

10. Find the cost of building a walk 5′ wide around a rectangular grass plot 100′ by 120′, the walk to be built on the plot. As a check, work in two ways.

11. The water reservoir on a range is 28″ long, 11″ wide and 12″ deep. It is to be lined with number 20 sheet copper which weighs 1.5855 pounds per square foot. Find the weight of copper required. Allow one half square foot for lapping and waste. Work two ways.

12.

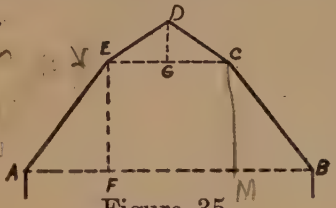


Figure 35.

A rectangular barn is 24′ wide, 36′ long and 16′ high to the eaves. The ends of the roof have the shape shown in the figure and the following dimensions AB=24′, AF=7′, EF=9′ 4″ and DG=3′ 4″. Find the cost of painting the barn at 90¢ per square (100 sq. ft.).

13. If the dimensions of the plot of ground represented by Figure 33 are as follows, find the area. FD=300′, EI=75′ FC=450′, DJ=150′, BK=125′, FB=250′ and AH=75′.

14. If the dimensions of Figure 34 are: FG=70 yd., GH=85 yd., HJ=100 yd., JI=25 yd., IC=70 yd., EG=65 yd., DI=75 yd., AH=95 yd., and BJ=85 yd., find the area.

162. Areas of Figures Bounded by Curved Lines. The only figure bounded by one or more curved lines the area of which can be obtained by elementary methods is the circle, which will be considered in the next section. The areas of other figures bounded by curved lines may, however, be approximated in various ways. One method that can sometimes be used is to trace the figure on cross section paper ruled in square inches. Then the area of the figure can be roughly estimated by counting the number of squares covered. (Figure 36).

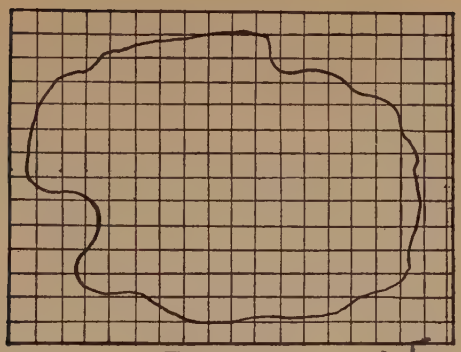


Figure 36.

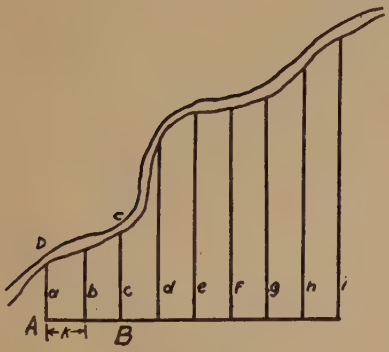


Figure 37.

Another method of finding the area bounded by a curved line is by *Simpson's Rule.* Suppose we wish to find the area of a portion of a field cut off by a winding stream such as is shown in Figure 37. The field is first divided into an *even* number of strips of equal width k, in this case there are 8 strips. First consider the double strip ABCD. This is shown on a larger scale in Figure 38. Divide the double strip into three strips of equal width. AE=EF=FB=$\frac{1}{3}\times$AB=$\frac{1}{3}\times 2k=\frac{2}{3}k$. Then the area of the double strip is approximately equal to the area of the trapezoid AEHD+the area of the rectangle EFGH+the area of the trapezoid FBCG or to $\frac{1}{2}\times\frac{2}{3}\ k\times(a+b)+\frac{2}{3}\ k\times b+\frac{1}{2}\times\frac{2}{3}k\times(b+c)=\frac{1}{3}\ ka+\frac{1}{3}\ kb+\frac{2}{3}\ kb+\frac{1}{3}\ kb+\frac{1}{3}\ kc=\frac{1}{3}\ ka+\frac{4}{3}\ kb+\frac{1}{3}\ kc=\frac{1}{3}k(a+4b+c)$.

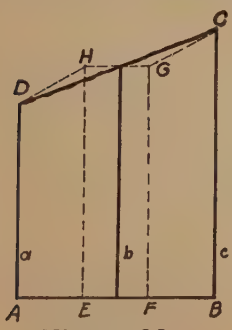


Figure 38.

Returning to Figure 37 we have:

Area 1st double strip$=\frac{1}{3}\ k\ (a+4b+c)$.

Area 2nd double strip$=\frac{1}{3}\ k\ (c+4d+e)$.

Area 3rd double strip$=\frac{1}{3}\ k\ (e+4f+g)$.

Area 4th double strip$=\frac{1}{3}\ k\ (g+4h+i)$.

Adding we get Simpson's Rule:

$$A=\frac{1}{3}\ k\ (a+4b+2c+4d+2e+4f+2g+4h+i).$$

III. The Measurement of the Circle.

163. Relation of Circumference to Diameter. By measuring the circumferences and diameters of circular objects of various sizes and dividing, the pupils can discover that the circumference of any circle is always just a little more than 3 times its diameter. The diameter of a small circular object can be measured accurately by placing it between two rectangular blocks and placing the blocks against the edge of a rule as shown in Figure 39. The distance between the blocks is evidently the diameter of the circle.

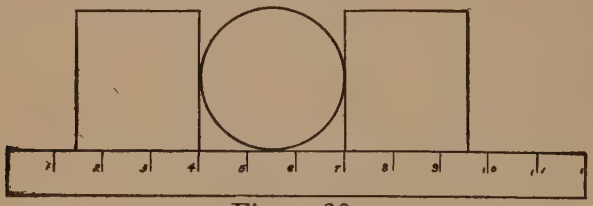


Figure 39.

The circumference of a small circular object can be measured in several ways. Perhaps the most accurate method is to wind a strip of paper tightly around the object so that the ends lap and with a pin make a small hole through both thicknesses of the paper at the lap. Take off the paper and measure the distance between the two pin holes, this distance equals the circumference of the object. If care is taken the circumference may be measured by marking a point on the circumference of the object, placing this mark over a mark on a piece of paper and care-

fully rolling the object until the mark on the circumference again comes to the bottom and marking the place where it touches the paper. The distance between the two marks on the paper is the circumference.

The circumference and diameter of larger objects such as buggy wheels can usually be measured directly with a tape line with a sufficient degree of accuracy.

When the pupils have discovered that the circumference is always a little more than three times the diameter they should be told that if they could take their measurements with sufficient accuracy they would find that $C=3\frac{1}{7}d$, or still more accurately that $C=3.1416\ d$. The symbol π, used to represent the number $3\frac{1}{7}$ or 3.1416 should also be given as the pupils may meet it later in practical formulas. The relation between the circumference and the diameter can then be stated:

The circumference of a circle is equal to π times the diameter, or $C=\pi\ d$.

The corresponding relation between the circumference and the radius should also be memorized.

The circumference of a circle is equal to $2\ \pi$ times its radius, or $C=2\ \pi\ r$.

The practical applications of these formulas should next be considered, namely they enable one to find the circumference when the radius or diameter are known or can be measured, and to find the radius or diameter when the circumference is known or can be measured. If one wishes to determine the diameter of a large smoke stack the easiest way is to measure the circumference and divide by π. If one wants to know the circumference of a circle drawn on the black board it is easier to measure the diameter and multiply by π than to measure the circumference directly.

164. The Area of the Circle. The formula for the area of the circle is obtained by changing the circle into a figure that approximates a parallelogram. Although the method used in obtaining the formula is an approximate one, the formula

Figure 40.

obtained is exact, as is proved in geometry. Each pupil should be supplied with a circle cut out of paper. By folding they can divide the circle into eight equal parts as shown in Figure 40. If the paper is cut along the creases and the parts rearranged as shown in Figure 41, the circle is changed into a figure resembling a parallelogram. The area of a parallelogram is the product of its base and altitude so the area of the figure in Figure 41 is equal to AB$\times r$. As nothing has been added to or taken away from the circle the area of the circle is also evidently AB$\times r$. But AB is one-half of the circumference and r is the radius of the circle (Figure 40). Therefore the area of a circle equals the product of one-half the circumference and the radius, or $A = \frac{1}{2}\ C \times r$.

Figure 41.

This formula for the area is not a convenient working formula as it would involve measuring both the circumference and the radius in order to find the area. The formula would be more convenient if it defined the area in terms of the radius alone. The circumference can be eliminated from the formula by using the relation $C = 2\pi r$. If this value is substituted for C in the formula $A = \frac{1}{2}C \times r$, we get $A = \frac{1}{2} \times 2\ \pi\ r \times r = \pi\ r^2$. Therefore:

The area of a circle is equal to π times the square of its radius, or $A = \pi\ r^2$.

The pupils should see that this formula can be used, not only to find the area when the radius is known or can be measured, but also to find the radius when the area is known.

EXERCISES AND PROBLEMS. IRREGULAR FIGURES. THE CIRCLE.

1. If each square in Figure 36 represents a square inch, estimate the area bounded by the curved line.

2. Draw an irregular closed curve on a large sheet of paper. Rule the paper in 1 inch squares and estimate the area bounded by the curve.

3. Find the number of acres in the portion of the field shown in Figure 37, if k=2 rd., a=3 rd., b=4 rd., c=4½ rd., d=5rd., e=7 rd., f=8½ rd., g=8¼ rd., h=8 rd., i=8¼ rd.

4. If possible, find the number of acres in some irregularly shaped tract of land by means of Simpson's Rule.

5. Measure the circumferences and diameters of several circular objects and compute the value π from each. Find the average of your results.

6. Draw a circle. Measure the diameter. Compute the circumference and the area.

7. If there is a pipe of any kind running through your school room measure its circumference with a tapeline or strip of paper and compute its diameter.

8. The size of sheets of tin is 20″ by 28″. How many circular discs for can bottoms each 4″ in diameter, can be cut from such a sheet? What per cent of tin is wasted? Draw a figure before trying to solve the problem.

9. How many sheets of tin of the above size will it take to make a gross of tin cans 3⅞″ in diameter and 4¾″ high, allowing ⅜″ on the diameter of the end pieces and ½″ on the length and 1″ on the height of the side pieces for fastening together? First draw a figure of an end and of a side piece, showing dimensions. Second draw a figure showing how many ends can be cut from one sheet. Next draw a figure showing how many side pieces can be cut from one sheet. How many sheets will it take? Find the per cent of waste.

10. A clothes boiler having parallel sides and semicircular ends is 14″ deep, 13″ wide and 21″ long. The sides are of tin and the bottom of copper. Find the number of square inches of tin and of copper in the boiler. Draw a figure showing the shape and dimensions of the bottom and a second figure show-

ing the shape and dimensions of the piece of tin forming the sides.

11. A rectangular sheet of $\frac{3}{8}''$ steel plate is $10''$ by $14''$. Four holes are drilled through the sheet, two $2\frac{1}{2}''$ in diameter each, and two $1''$ in diameter each. Find the weight of the sheet if $\frac{3}{8}''$ steel plate weighs 15.3 pounds per square foot.

12. A tin ring for the bottom of a cake pan is $8''$ in diameter and has a hole in the center $2''$ in diameter. Find the number of square inches of tin.

13. A horse is tethered at one corner of a barn that is $24'$ by $26'$, by a rope $20'$ in length. Draw a figure showing the barn and the area the horse can graze over. Find the area that he can graze over.

14. If the rope by which the horse in the above problem is tethered is increased to 30 ft. in length, over how many square feet can he graze? Draw a figure.

15. If the rope is increased to 40 feet in length over how much space can the horse graze? Draw a figure.

16. How far will an automobile wheel that is 34 inches in diameter move forward in making one revolution? Make a general statement of how far any wheel moves forward in one revolution.

17. How many revolutions does a 32 inch automobile wheel make in going 1 mile?

18. How many revolutions per minute (R. P. M.) does a 34 inch automobile wheel make when the car is going 30 miles an hour?

19. How fast is the automobile going when the wheels are making 300 R. P. M., if the wheels are 32 inches in diameter?

20. Grindstones when turned by power at too high a speed tend to fly apart or "burst." It has been found by experiment that a grindstome made of a certain kind of stone can safely stand a rim speed of 3600 feet per minute. Through how many revolutions per minute is it safe to turn a grindstone 4 feet in

diameter and made of this material?

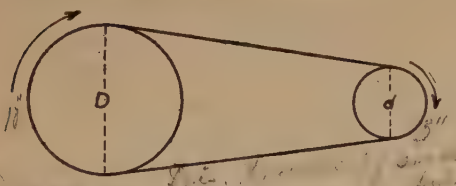


Figure 42.

21. Figure 42 shows two pulleys connected by a belt. D represents the diameter of the large pulley, d the diameter of the small one, s the speed or R. P. M. of the large pulley, S the speed or R. P. M. of the small pulley. A point on the rim of the large pulley moves in one revolution a distance equal to the circumference of the pulley or πD, and in one minute or s revolutions a distance equal to $s \times \pi D$. In the same time the small pulley makes S revolutions and a point on its rim moves a distance $S \times \pi d$. But since the two pulleys are connected by a belt a point on the rim of one moves the same distance as a point on the rim of the other, so $s \times \pi D = S \times \pi d$, or $s D = S d$. This is known as the *Law of Pulleys* and is used in the following problems. The student should state this law in words.

22. The drive wheel of a certain steam engine is 10 feet in diameter and is making 100 R. P. M. It is belted to a pulley 2′ 6″ in diameter. How many R. P. M's. is the pulley making? Draw a figure.

23. A farmer buys a gasoline engine to run a cream separator and a washing machine. Instead of connecting the engine directly to the separator and washer, he connects it to an overhead line-shaft and runs the separator and washer from the shaft. The driving pulley on the engine is 4″ in diameter and makes 600 R. P. M. He belts this to a 12″ pulley on the line-shaft. How many R. P. M. does the line-shaft make?

24. The cream separator mentioned in the preceding problem is supposed to be turned at the rate of 50 R. P. M. If the pulley on the separator is 12″ in diameter, what size pulley will the farmer have to use on the line-shaft to drive the separator?

25. The washing machine mentioned in Problem 23 is supposed to be turned at the rate of 105 R. P. M. and the pulley on the machine is 18″ in diameter. What size pulley must the farmer use on the line-shaft to drive the washer?

IV. Square and Cube Root.

165. Definitions. If a number is the product of two equal factors, it is called the *square* or *second power* of one of them. Thus $9=3\times3$, so 9 is the square of 3. This is written $9=3^2$. The 2 above and to the right of the 3 is called an *exponent* and indicates that 3 is to be used as a factor 2 times. A number that is the product of three equal factors is called the *cube* or *third power* of one of them. Thus $8=2\times2\times2$, so 8 is the cube of 2. This is written $8=2^3$.

If a number is the product of two equal factors, one of these factors is called the *square root* of the number. Thus $64=8\times8$, so by definition 8 is the square root of 64. This is written $8=\sqrt{64}$. If a number is the product of three equal factors, one of these factors is called the *cube root* of the number. Thus $27=3\times3\times3$, so 3 is the cube root of 27. This is written $3=\sqrt[3]{27}$.

Pupils in the elementary school, in connection with the work in practical measurement, often need to know the square and cube roots of numbers.

166. Methods of Finding Square Root. Three methods of finding square roots will be considered (1) square roots of perfect squares by factoring, (2) approximate square roots by the "trial and error" method, (3) approximate square roots by means of tables. Two methods of finding cube roots are given, (1) by factoring, and (2) by tables. The ordinary methods of extracting approximate square and cube roots have little practical value and need not be taught in the elementary schools.

167. Square Roots of Perfect Squares by Factoring. In the case of perfect squares the square roots can easily be obtained by factoring. Thus $36=2\times2\times3\times3=(2\times3)\times(2\times3)=6\times6$, so $\sqrt{36}=6$. Some practice of this kind should be given to emphasize the meaning of square root. The squares of all of the numbers from 1 to 15 inclusive and the square roots of the corresponding perfect squares should be memorized. Tables of the squares of the numbers at least as high as 32^2 should be available for reference. If the book does not give such a table the pupils should make their own.

168. Approximate Square Roots by "Trial and Error". The practical method of finding approximate square roots of numbers that are not perfect squares is by means of tables such as that given and explained in the next section. If tables are not available approximate square roots may be found by guessing and correcting the guess.

Example. Find $\sqrt{69}$.

Solution. $8^2=64$ and $9^2=81$, so $\sqrt{69}$ is between 8 and 9 but nearer 8 than 9. Suppose we try 8.2. As this proves to be too small we try 8.3 and 8.4.

8.2	8.3	8.4
8.2	8.3	8.4
164	249	336
656	664	672
67.24	68.89	70.56

From the above it is evident that $\sqrt{69}=8.3$ to the nearest tenth.

In this manner the approximate square roots of integers up to 1000 can easily be found to the nearest tenth which is accurate enough for practical purposes.

169. Cube Root of Perfect Cubes by Factoring. The cube root of a perfect cube may be found by factoring. Thus $216=2\times2\times2\times3\times3\times3=(2\times3)\times(2\times3)\times(2\times3)$. Therefore $\sqrt[3]{216}=6$. The cubes of the numbers from 1 to 10 should be memorized. A table giving the cubes of larger numbers should also be available for reference.

170. Approximate Square and Cube Roots by Tables. Any one who frequently has occasion to find square or cube roots makes use of a table such as that given on the following page. The squares, cubes, square roots and cube roots of integers from 1 to 100 may be read directly from the table.

This table may also be used to find the roots of decimals.

Example 1. Find $\sqrt{87.4}$.

Solution.

$\sqrt{88}=9.381$
$\sqrt{87}=9.327$
$.054$
$.4$
$.0216$
9.327
9.349
$\sqrt{87.4}=9.349$ (Approx.)

Explanation. As 87.4 lies between 87 and 88 the $\sqrt{87.4}$ is between $\sqrt{87}$ and $\sqrt{88}$. 88 is 1 more than 87 and $\sqrt{88}$ is .054 more than the $\sqrt{87}$. 87.4 is .4 more than 87 so the $\sqrt{87.4}$ will be approximately $.4\times.054$ more than $\sqrt{87}$. The table gives the root to the nearest thousandth so the product $.4\times.054=.0216$ is also used only to the nearest thousandth, namely .022.

Example 2. Find $\sqrt{69.32}$

Solution. 69.32 is between 69 and 70. Notice the difference between $\sqrt{69}$ and $\sqrt{70}$ which is .06. Take $.32\times.06$ using product to the nearest thousandth, namely .019. Add .019 to the $\sqrt{69}$. $\sqrt{69.32}=8.326$.

Example 3. Find $\sqrt[3]{97.2}$

Solution. 97.2 is between 97 and 98. Notice the difference between the $\sqrt[3]{97}$ and the $\sqrt[3]{98}$ which is .015. Take $.2\times.015$, using the product to the nearest thousandth, namely .003. Add .003 to the $\sqrt[3]{97}$. $\sqrt[3]{97.2}=4.598$.

Table of Squares, Cubes, Square Roots and Cube Roots. 1 to 100.

No.	Sq.	Cube.	Square Root.	Cube Root.	No.	Sq.	Cube.	Square Root.	Cube Root.
1	1	1	1.000	1.000	51	2601	132651	7.141	3.708
2	4	8	1.414	1.260	52	2704	140608	7.211	3.733
3	9	27	1.732	1.442	53	2809	148877	7.280	3.756
4	16	64	2.000	1.587	54	2916	157464	7.348	3.780
5	25	125	2.236	1.710	55	3025	166375	7.416	3.803
6	36	216	2.449	1.817	56	3136	175616	7.483	3.826
7	49	343	2.646	1.913	57	3249	185193	7.550	3.849
8	64	512	2.828	2.000	58	3364	195112	7.616	3.871
9	81	729	3.000	2.080	59	3481	205379	7.681	3.893
10	100	1000	3.162	2.154	60	3600	216000	7.746	3.915
11	121	1331	3.317	2.224	61	3721	226981	7.810	3.936
12	144	1728	3.464	2.289	62	3844	238328	7.874	3.958
13	169	2197	3.606	2.351	63	3969	250047	7.937	3.979
14	196	2744	3.742	2.410	64	4096	262144	8.000	4.000
15	225	3375	3.873	2.466	65	4225	274625	8.062	4.021
16	256	4096	4.000	2.520	66	4356	287496	8.124	4.041
17	289	4913	4.123	2.571	67	4489	300763	8.185	4.062
18	324	5832	4.243	2.621	68	4624	314432	8.246	4.082
19	361	6859	4.359	2.668	69	4761	328509	8.307	4.102
20	400	8000	4.472	2.714	70	4900	343000	8.367	4.121
21	441	9261	4.583	2.759	71	5041	357911	8.426	4.141
22	484	10648	4.690	2.802	72	5184	373248	8.485	4.160
23	529	12167	4.796	2.844	73	5320	389017	8.544	4.179
24	576	13824	4.899	2.884	74	5476	405224	8.602	4.198
25	625	15625	5.000	2.924	75	5625	421875	8.660	4.217
26	676	17576	5.099	2.962	76	5776	438976	8.718	4.236
27	729	19683	5.196	3.000	77	5929	456533	8.775	4.254
28	784	21952	5.292	3.037	78	6084	474552	8.832	4.273
29	841	24389	5.385	3.072	79	6241	493039	8.888	4.291
30	900	27000	5.477	3.107	80	6400	512000	8.944	4.309
31	961	29791	5.568	3.141	81	6561	531441	9.000	4.327
32	1024	32768	5.657	3.175	82	6724	551368	9.055	4.344
33	1089	35937	5.745	3.208	83	6889	571787	9.110	4.362
34	1156	39304	5.831	3.240	84	7056	592704	9.165	4.380
35	1225	42875	5.916	3.271	85	7225	614125	9.220	4.397
36	1296	46656	6.000	3.302	86	7396	636056	9.274	4.414
37	1369	50653	6.083	3.332	87	7569	658503	9.327	4.431
38	1444	54872	6.164	3.362	88	7744	681472	9.381	4.448
39	1521	59319	6.245	3.391	89	7921	704969	9.434	4.465
40	1600	64000	6.325	3.420	90	8100	729000	9.487	4.481
41	1681	68921	6.403	3.448	91	8281	753571	9.539	4.498
42	1764	74088	6.481	3.476	92	8464	778688	9.592	4.514
43	1849	79507	6.557	3.503	93	8649	804357	9.644	4.531
44	1936	85184	6.633	3.530	94	8836	830584	9.695	4.547
45	2025	91125	6.708	3.557	95	9025	857375	9.747	4.563
46	2116	97336	6.782	3.583	96	9216	884736	9.798	4.579
47	2209	103823	6.856	3.609	97	9409	912673	9.849	4.595
48	2304	110592	6.928	3.634	98	9604	941192	9.899	4.610
49	2401	117649	7.000	3.659	99	9801	970299	9.950	4.626
50	2500	125000	7.071	3.684	100	10000	1000000	10.000	4.642

This table may also be used to find the square roots of numbers greater than 100 and decimals less than 1. To do this we make use of the following. Since $\sqrt{100}=10$, multiplying

or dividing a number by 100 multiplies or divides the square root of the number by 10. Or, stating the same thing in another way, shifting the decimal point two places to the right or left in the number shifts the point only one place in the square root of the number. We make use of this principle to find the square roots of numbers outside the range of the table, that is less than 1 or more than 100.

Example 1. Find $\sqrt{723}$.

Solution and Explanation. Mentally shift the point *two* places to the left (divide by 100) to bring the number between 1 and 100, the limits of our table. This makes the number 7.23. Notice the difference between $\sqrt{7}$ and the $\sqrt{8}$ which is .182. $.23\times.182=.042$ to the nearest thousandth. Add .042 to the $\sqrt{7}$, giving 2.688. In the beginning we divided 723 by 100 which divided the $\sqrt{723}$ by 10. So the result obtained above must be multiplied by 10 or the point shifted one place to the right. Therefore $\sqrt{723}=26.88$.

Example 2. Find $\sqrt{.593}$.

Solution. Mentally shift the point two places to the right (multiply by 100). This makes the number 59.3 and so brings it within the range of the table. Notice the difference between the $\sqrt{59}$ and the $\sqrt{60}$ which is .065. Take $.3\times.065=.020$ to the nearest thousandth. Add .020 to the $\sqrt{59}$ which gives 7.701. In the beginning we multiplied the number by 100 which multiplied the root by 10 so the 7.701 must be divided by 10 or the point shifted one place to the left. Therefore $\sqrt{.593}=.7701$.

Exercise. Square and Cube Root.

1. Find the following to the nearest tenth by the Trial and Error method.

1. $\sqrt{39}$
2. $\sqrt{7}$
3. $\sqrt{78}$
4. $\sqrt{57}$

2. Find the following using the table.

1.	$\sqrt{95}$	11.	$\sqrt{159}$
2.	$\sqrt{53}$	12.	$\sqrt{347}$
3.	$\sqrt[3]{71}$	13.	$\sqrt{1859}$
4.	$\sqrt[3]{89}$	14.	$\sqrt{5684}$
5.	$\sqrt{72.8}$	15.	$\sqrt{.509}$
6.	$\sqrt{17.25}$	16.	$\sqrt{.279}$
7.	$\sqrt{83.37}$	17.	$\sqrt{.7534}$
8.	$\sqrt[3]{91.4}$	18.	$\sqrt{.6571}$
9.	$\sqrt[3]{37.5}$	19.	$\sqrt{.0375}$
10.	$\sqrt{873}$	20.	$\sqrt{.0059}$

V. The Pythagorean Theorem.

171. History. One of the most important geometrical relations, from the practical point of view, is that which exists between the three sides of any right triangle, namely that the square on the hyoptenuse of any right triangle is equal to the sum of the squares on the other two sides. Or, in Figure 43, $b^2=a^2+c^2$.

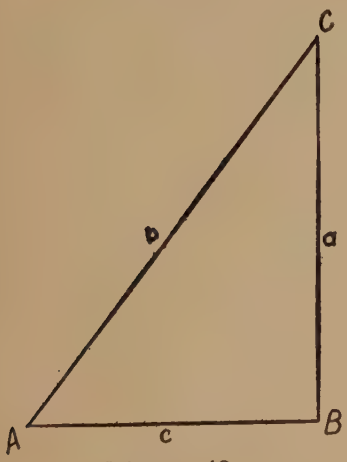


Figure 43.

This relationship has been known for a long time and was probably first discovered by the Egyptians at least for certain special cases. It is not known how the Egyptians made the discovery but it is probable that it was by observing the arrangement of tiles in some of their tile floors. Figure 44 shows how the theorem might have been discovered. The figure represents

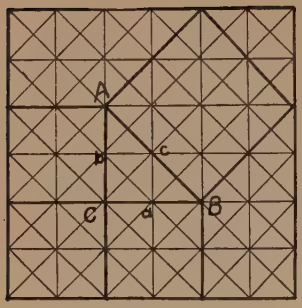


Figure 44.

a portion of a tile floor composed of small right angled triangles. Certain lines are made heavier in the figure to aid the eye. In the right triangle ABC, the square on the hypotenuse c contains 32 small triangular tiles, and each of the squares on the other two sides a and b contains 16 small tiles. It is therefore seen that $c^2=a^2+b^2$.

The Egyptians made use of this theorem in laying off right angles in their surveying. Their method was probably similar to the following method which may be used by the pupils in the grades to lay off right angles in their field work. Fasten the ends of a rope 12 feet long together so as to form a continuous ring. Mark three points on the rope 3, 4 and 5 feet apart, respectively. Now if three pupils grasp the rope at the points marked and stretch it a triangle will be formed having sides 3 feet, 4 feet, and 5 feet long. This will be a right triangle because $3^2+4^2=5^2$.

Although this theorem was discovered, at least for special cases, by the Egyptians its general truth was first proved by the Greeks. The first proof of the theorem was probably due to Pythagoras or some member of his school about 500 B. C. For this reason the theorem is usually called the Pythagorean Theorem.

172. Verification of the Theorem in the Elementary School. The truth of the Pythagorean Theorem can be verified in the elementary schools in a variety of ways. Figures similar to Figure 44 can be used for this purpose. A very interesting method is shown in Figures 45 and 46.

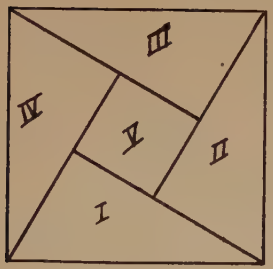


Figure 45.

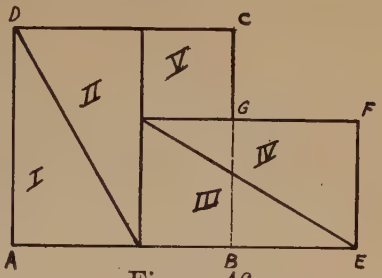


Figure 46.

The pupils can cut a large square out of cardboard and cut it up into four equal right triangles and a small square as shown in Figure 45. As arranged in Figure 45 the cardboard forms the square on the hypotenuse of one of the four equal right triangles. The parts may then be rearranged, as shown in Figure 46, so as to become the sum of two squares ABCD and BEFG. But ABCD is the square on one side of one of the four equal right triangles and BEFG is the square on the other side of another of the equal right triangles. So the square on the hypotenuse can be rearranged so as to form the sum of the squares on the other two sides of the right triangle. This method of establishing the truth of the theorem is due to the Hindu mathematician Bhaskara (born 1114 A. D.).

The method of dividing the square into four equal right triangles and a square, as shown in Figure 45, is shown in Figure 47. First draw a line as AB. Then draw CD perpendicular to AB, EF perpendicular to CD, and GH perpendicular to EF. Erase IB.

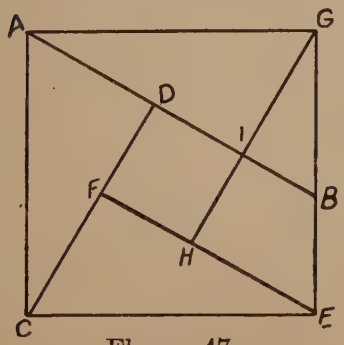


Figure 47.

The truth of the theorem can also be established through measurement. If the three sides of a number of right triangles are accurately measured and squared it will be found that the square of the hypotenuse in each case is ap-

proximately equal to the sum of the squares of the other two sides.

Exercise. Pythagorean Theorem.

1. Cut out of cardboard the figure shown in Figures 45 and 47, and with this model prove the Pythagorean proposition.

2. Starting at the corner of a room measure a distance of 3 feet along one side from the corner and 4 feet along the other side. Measure the distance AB between the ends of these two distances as shown in the figure. Is the angle of the floor, C, a right angle? Why?

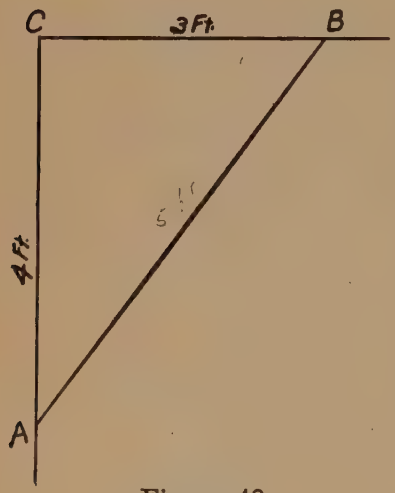


Figure 48.

3. How could you determine whether or not a fence post is perpendicular to the plane of the ground, using only a tape measure?

Note: A line is perpendicular to a plane if it is perpendicular to two different lines lying in that plane.

4. If c represents the hypotenuse of a right triangle and a and b the other two sides, find:

1. c, if a=7 ft., b=8 ft.
2. c, if a=34 ft., and b=23 ft.
3. a, if c=21.2 ft., and b=11.7 ft.
4. b, if c=117 ft., and a=72 ft.

5. If h, w and l represent the height, width and length respectively of a rectangular room show that the formula $d=\sqrt{l^2+w^2+h^2}$ is the formula for finding the length of a diagonal or the distance from an upper corner of the room to the diagonally opposite lower corner.

6. Use this formula to find the diagonal of a room 10′x12′x9′.

7. The left hand figure, in Figure 49, shows the shape and dimensions of a candle shade to be made of parchment. The right hand figure shows the pattern to be used in cutting the parchment. In order to be able to make this pattern the following must be known AB, BO and the length of the arc AC. AB and BO can be found by means of similar triangles and the Pythagorean Theorem. How can the length of the arc be found? Find AB, BO and AC. Make the pattern out of heavy paper and make it into the form of a shade. As a check measure the shade to see if it has the proper dimensions.

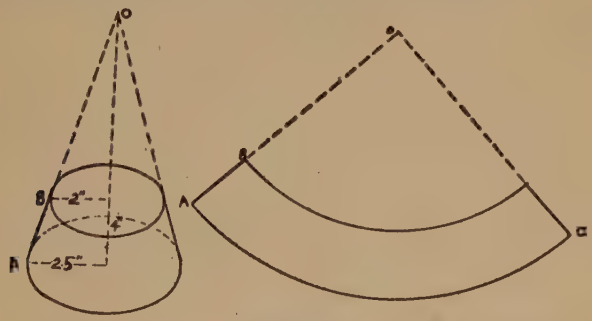


Figure 49.

VI. Volumes and Surfaces.

173. Volumes of Rectangular Prisms. The unit usually used in measuring volumes is the volume of a cube whose edge is one linear unit. The volume of a cube whose edge is 1 inch is called 1 *cubic inch,* the volume of a cube whose edge is one foot is called 1 *cubic foot,* etc.

It has already been seen that the only surface the area of which can be found by direct measurement, that is by applying the unit of measure, is the rectangle. The only solid the volume of which can be found by direct measurement is

the rectangular prism. To develop the rule for the volume of a rectangular solid the pupils should build up such a solid using cubical blocks one inch on a side, as shown in Figure 50. The volume, or the number of cubic inches in such a solid can first be found by counting the number of cubic inches of which it is composed. A shorter method is to count the number of cubic inches in one layer (the top layer is most convenient) and multiply this by the number of layers. Each cubic inch of the bottom layer covers one square inch of the base and as the entire base is covered there are evidently as many cubic inches in one layer as there are square inches in the base. Each layer is one inch high so there are always just as many layers as there are inches in the altitude of the solid. It follows, therefore, that the number of cubic inches in the solid is equal to the product of the number of square inches in the base and the number of inches in the altitude, or

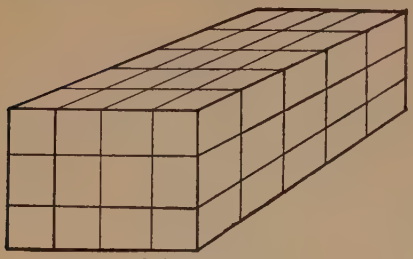


Figure 50.

The volume of a rectangular prism is equal to the product of its base and altitude, or $V=b\times a$.

As the area of the base is found by multiplying the length of the solid by its width it follows that

The volume of a rectangular solid is equal to the product of its three dimensions, or $V=l\times w\times h$.

As all three dimensions of a cube are equal it follows that:

The volume of a cube is equal to the cube of one of its sides, or $V=s^3$.

174. Volume of Right Triangular Prisms. If a rectangular prism is cut into two parts, as shown in Figure 51,

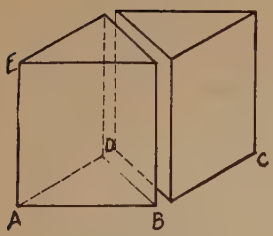


Figure 51.

two equal triangular prisms are formed. The volume of the rectangular prism is the area ABCD$\times$AE, but the triangular prism ABDE is just half the rectangular prism, or one-half the area ABCD$\times$AE. But one-half the area of ABCD is the area ABD or the area of the base of the triangular prism ABD-E, and AE is the altitude of the triangular prism, Therefore,

The volume of a triangular prism equals the product of the area of its base and its altitude, or $V=b\times a$.

175. Volume of Any Right Prism. Any right prism may be divided into a number of triangular prisms as shown in Figure 52. The volume of the first triangular prism is its base b_1 times its altitude a. The volume of the second triangular prism is its base b_2 times its altitude a, and the volume of the third triangular prism is its base b_3 times its altitude a. The total volume of the prism is the sum of the volumes of the triangular prisms composing it, or $b_1a+b_2a+b_3a=(b_1+b_2+b_3)a$. But $b_1+b_2+b_3$ is the base of the prism and a is the altitude, therefore:

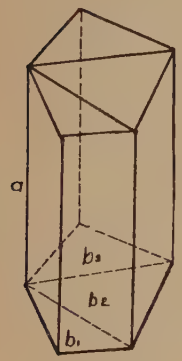


Figure 52.

The volume of any right prism is equal to the product of its base and altitude.

176. Surface of Right Prism. Finding the surfaces of right prisms involves no new theory as the lateral surfaces are all rectangles and the bases are polygons.

177. Volume of a Right Circular Cylinder. The simplest form of cylinder is one whose bases are circles and whose axis, or line joining the centers of the two bases, is perpendicular to the bases. Such a cylinder is called a *right circular cylinder.*

Such a cylinder can be changed into a solid that approximates a prism as shown in Figure 53. The volume of the prism is the area ABCD times the altitude *a*. But the volume of the cylinder equals the volume of the prism as nothing has been added or taken away, so the volume of the cylinder is the area ABCD times *a*. But the area ABCD equals the area of the base of the cylinder and *a* is the altitude of the cylinder. Therefore:

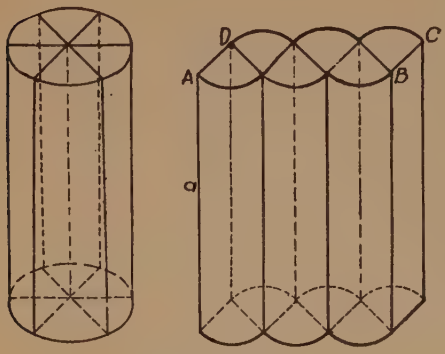


Figure 53.

The volume of a cylinder is equal to the product of its base and altitude, or $V=b\times a$.

178. Generalization. The method of finding the volume of a solid by multiplying the area of the base by the altitude is very general in its application and applies not only to right prisms and cylinders but to slant ones as well. In fact it is true for any solid having two parallel equal bases and a uniform cross section parallel to the bases. This cannot be proven by elementary means but should be known because of its practical importance.

179. Surface of Cylinder. The surface of a right circular cylinder may be easily found. The two bases are circles and if the lateral surface is covered with a piece of paper cut to exactly fit it and the paper is taken off of the cylinder and laid flat it is seen to be a rectangle with altitude equal to the altitude of the cylinder and base equal to the circumference of the base of the cylinder. Therefore:

The lateral surface of a right circular cylinder is equal to the product of the altitude and the circumference of the base of the cylinder or equal to $2\pi ra$ *where* a *is the altitude and* r *the radius of the base.*

EXERCISES AND PROBLEMS. PRISMS AND CYLINDERS.

1. If possible secure models and develop in detail, without consulting the book, the rules for finding the:

a. Volume of a rectangular solid.
b. Volume of a triangular prism.
c. Volume of any prism.
d. Volume and lateral surface of a cylinder.

2. If possible measure several objects in the form of prisms or cylinders and determine their volumes and surfaces.

3. A farmer has a wooden watering trough 10′ long, 28″ wide at the top, 20″ wide at the bottom and 24″ deep, inside measurements. How many gallons of water will it hold. (1 cu. ft.=7½ gal. approx.) Draw a figure.

4. A farmer makes a V-shaped trough to hold water for his chickens. It is 8′ long, 10″ across the top and 5″ deep, inside measurements. How many gallons will it hold? (231 cu. in. =1 gal.) Draw a figure.

5. A certain refrigerator will hold a piece of ice 11″ by 13″ by 17″. How many pounds will it hold? (1 cu. ft. of ice weighs 56.25 pounds).

6. A coal bin is 15′ long and 12′ wide and can be filled to a depth of 7′. How many tons of coal will it hold? (1 cu. ft. weighs 63 lbs.)

7. It has never taken over 8 tons of coal to heat a certain house. The owner wishes to build a bin in the cellar large enough to hold this amount. The cellar is only 7′ deep. How many square feet will there have to be in the base of the bin?

8. A farmer has a corn crib 25′ long and 10′ wide. It is filled by throwing the corn in at a row of windows along the top of one of the long sides. If the corn stands 9′ deep under the windows and slopes off to 7′ deep on the opposite side, how many bushels does the crib contain? (Corn on the ear averages about $\frac{2}{5}$ bushel to 1 cu. ft.) Draw a figure.

9. A railway cuts through a hill. The cut is an eighth of a mile long and averages 10′ deep, 70′ wide at the top and 40′ wide at the bottom. How many cubic yards were removed in making the excavation? Draw a figure.

10. The dirt and rock removed from the cut were used to make a fill across a low place. The fill is 500′ long, 30′ wide at the top, 40′ wide at the bottom, and $3\frac{1}{2}$′ high. How many cubic yards did it take to make the fill? Draw a figure.

11. A wooden flume is used to bring water from a stream to a factory. The flume is 6′ wide. How many gallons of water is the flume carrying to the factory per hour when the water is running at a rate of 80′ per minute and to a depth of 3′?

12. A boy in a manual training class turns a wooden cylinder 4″ in diameter and 8″ long out of a block $4\frac{1}{2}$″ in diameter and 9″ long. How much wood is cut away in shavings? What per cent?

13. A bushel measure is to be made of galvanized iron. If it is made 16″ in diameter, how deep will it have to be? How many square inches of metal are required? Allow 1″ on the diameter of the bottom for fastening, 1″ on length of side piece for seam, and 2″ on width of side piece for fastening at bottom and roll at top. (1 bu.=2150 cu. in.)

14. How many gallons of water will the clothes boiler described in Problem 10, page 199 hold? Draw a figure.

15. A man wants to dig a cistern large enough to hold 35 barrels. Owing to the presence of rock, it can only be dug 5′ deep. What must the diameter be? (1 barrel=$31\frac{1}{2}$ gal.)

16. A village water works system uses a stand pipe 16′ in diameter to maintain pressure. When the pipe is filled to a depth of 30′, what is the pressure on the bottom of the tank in pounds per square inch? (1 cu. ft. water weighs 62.5 pounds).

17. In order to find out how much water there is in a cistern, a boy lets down a weight on the end of a string until it touches the bottom. He finds that the string is wet 2′ 9″. How many gallons of water are there in the cistern if it is 7′ in diameter?

18. A horizontal cylindrical boiler 5′ in diameter and 18′ long is traversed from end to end by 70 pipes or flues, 2″ in diameter, which carry the heat from the fire through the water. How many gallons of water will the boiler hold? How much heating surface do the flues apply to the water?

19. Cylindrical iron window weights are made 1½″ in diameter. They must be of different weights for different windows. How long must a weight be made to weigh 8 lbs.? (Cast iron weighs .26 lbs. per cubic inch).

20. A lawn roller is made in the form of a hollow iron cylinder 1.5′ in diameter and 2′ long and weighs empty 50 lbs. Its weight can be increased by filling with water. To what depth (when standing on end), must it be filled in order to weigh 200 pounds?

21. Water runs through a pipe 12″ in diameter at the rate of 40′ per minute. How many gallons are carried by the pipe in 24 hours?

22. A pump makes 20 strokes a minute in each direction and pumps water on each stroke, forward and backward. The piston is 10″ in diameter and makes a stroke of 18″. How many gallons of water will it pump an hour?

23. How many tons of hay can be stored in the barn described in Problem 12, page 194, if the floor of the hay mow is 8′ from the floor of the barn at each side and 16′ from the floor

of the barn over a central driveway 8′ wide, running lengthwise through the barn. Draw a figure. (Allow 512 cu. ft. of hay to the ton).

24. Find the weight of a brass tube 12′ long, inner diameter 1″ and outer diameter 1½″. (1 cu. in. brass weighs .3 lb.)

25. Which will hold more, a cylindrical tank 12′ long and 6′ in diameter or two tanks each 12′ long and 3′ in diameter? How much more? Can you answer these questions without finding the capacity of any of the tanks?

26. A farmer has a herd of 20 cows. He wants to store enough silage to provide 30 pounds per head per day for 180 days. If he builds a silo 14′ in diameter, how high must it be? (1 cu. ft. of silage weighs 30 lbs.)

27. The diameter of the silo built depends upon the number of cattle to be fed as the silage spoils when exposed to the air for several days. If it takes 10 cows to keep the silage from spoiling in a 10′ silo, how many will be necessary to keep it from spoiling in a 20′ silo?

28. If a farmer feeds 18 head cattle, what would be the largest diameter he could build his silo to prevent spoiling under the conditions stated in the preceding problem?

29. An instrument for measuring the amount of rainfall is shown in the figure. It is called a rain gauge. By gathering the water over a large area and storing it over a small area it increases the depth and makes possible more accurate measurements. If the diameter of the top is 12″ and of the bottom 4″, how much deeper will the same amount of water stand in the gauge than in a straight sided cylindrical vessel 12″ in diameter? If after a rain the water stands 4½ inches deep in the gauge, what was the depth of the rainfall on level ground? State a general rule for finding the depth of rainfall on level ground from the depth in the gauge.

Figure 54.

30. Rain from a flat tin roof 20′ by 30′ is conducted into a cistern 8′ in diameter. If the depth of water in the cistern is increased 4′ by a rain, how many inches of rain fell on level ground?

31. A rain of .42 in on level ground would increase the depth of water in the cistern of Problem 30 by what amount?

32. How many tons of water fall on an acre of land in a rainfall of .42 in.?

180. Volume of a Pyramid. The method of finding the volume of a pyramid is most easily discovered by comparing the pyramid to a prism having the same base and the same altitude. Hollow models made of metal or cardboard should be used. If the hollow pyramid is filled with sand or water and emptied into the prism, filled again and emptied, and so on until the prism is full, it will be found that it takes three of the pyramids to equal the prism. The volume of the prism is the product of its base and altitude and as the pyramid has the same base and altitude but its volume is just one-third that of the prism, it follows that:

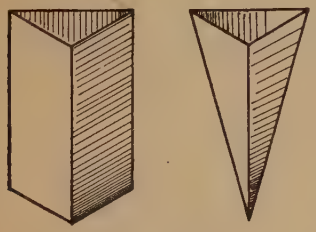


Figure 55.

The volume of a pyramid is equal to one-third the product of its base and altitude, or $V = \frac{1}{3}\ b \times a$.

181. Surface of Pyramid. Since the base of any pyramid is always a polygon and the other faces are triangles the surface of any pyramid can be found by taking the proper measurements on each face, finding the areas of the faces separately and adding. In the case of the regular pyramid the work of finding the lateral surface may be shortened. A *regular pyramid* is one whose base is a regular polygon (all angles and sides equal) and whose altitude goes through the center of the base. The

altitudes of the triangular faces of such a pyramid are all equal and any one of them is called the *slant height* of the pyramid.

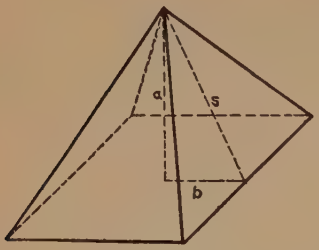


Figure 56.

In Figure 56, a is the altitude of the pyramid and s is the slant height. It should be noticed that a, s and the line b form a right triangle so that if any two of them are known the third can be found by the Pythagorean theorem.

It is evident from Figure 56, that:

The lateral surface of a regular pyramid is equal to one-half the product of the slant height and the perimeter of the base.

The proof of this is left to the student as an exercise.

182. The Volume of the Cone. The method of finding the volume of a cone can be determined by comparing the cone to a cylinder having the same base and altitude. Hollow models such as shown in Figure 57 should be used. The proof of the following rule is similar to that suggested for the volume of the pyramid and is left to the student as an exercise.

Figure 57.

The volume of a cone is equal to one-third the product of its base and altitude, or $V=\frac{1}{3}b\times a$.

183. The Surface of a Cone. The simplest type of cone is the *right circular cone.* In such a cone the base is a circle and the altitude goes through the center of the base. The altitude, slant height and the radius of the base form a right triangle so if any two of these are known the

third can be found. If the lateral surface of such a cone is covered with a piece of paper cut to fit and the paper is then laid out flat it is found to be a section of a circle as shown in Figure 58. The area of such a figure is found in the same way as the area of a triangle, that is the area of ABC is equal to one-half the product of AB and s. But AB is the circumference of the base of the cone and s is the slant height, therefore:

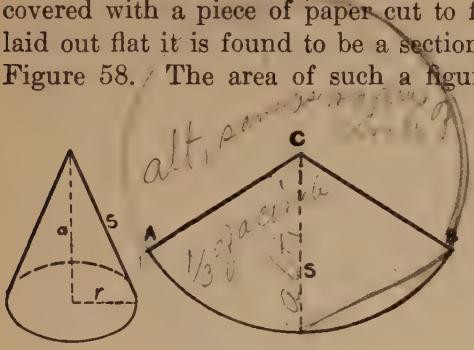


Figure 58.

The lateral surface of a right circular cone is equal to half the product of the circumference of the base and the slant height, or $L = \frac{1}{2} \times 2\pi r \times s$.

EXERCISES AND PROBLEMS. PYRAMIDS AND CONES.

1. Prove the rule given on Page 220 for finding the lateral surface of a regular pyramid.

2. If possible secure or make the hollow models and prove the rules for finding the volumes of pyramids and cones.

3. A regular square pyramid has a base 4″ by 4″ and is 3″ high. Draw the figure and find:

   a. The volume.
   b. The slant height.
   c. The lateral and total surfaces.

4. A regular six sided pyramid is 8″ high and each side of the base is 3″. Draw a figure and find the same things as in Problem 3.

5. A right circular cone is 8″ high and the diameter of the base is 5″. Draw a figure and find the same things as in Problem 3.

6. A right circular cone has a base 10″ in diameter and a slant height of 12″. Draw the figure and find:

a. The lateral and total surfaces.
b. The altitude.
c. The volume.

7. If possible, measure objects in the form of pyramids or cones and find their volumes and surfaces.

8. A farmer has a conical shaped pile of wheat on his granary floor. The slant height is 5′ and the diameter of the base is 10′. How many bushels are in the pile? Draw a figure. (1 bu.=1¼ cu. ft. approx.)

9. A farmer has a pile of oats in the corner of a bin in the shape of a fourth of a cone. The slant height is 6 ft. and the distance around the front of the pile from one wall to the other (one fourth the circumference of the base of the cone) is 6.5 ft. How many bushels are there in the pile? Draw a figure.

10. A man has a pile of hard coal in his cellar. It is piled against a wall in the shape of a half cone. It reaches on the wall to a height of 6′ and extends out from the wall on the floor a distance of 4.5′. How many tons of coal are in the pile? draw a figure. (36 cu. ft. of hard coal weighs 1 ton).

11. The portion of a cone or pyramid cut off between the base and a plane parallel to the base is called a frustrum. The formula for the lateral surface of the frustrum of a right circular cone or regular pyramid is $l=\frac{1}{2}s\times(p+p')$ where s is the slant height and p and p' are the perimeters of the two bases. Derive this formula in the case of the frustrum of a regular pyramid.

12. Use the formula given in the preceding problem to find the number of square inches of tin required to make a bucket 10½″ in diameter across the top, 8½″ in diameter at the bottom and 9½″ in slant height. Draw a figure.

13. The formula for the volume of the frustrum of a pyramid or cone is $V=\frac{1}{3}\ a\ (b+\sqrt{bb'}+b')$ where a is the altitude of the frustrum and b and b' are the areas of the two bases. Derive this formula in the case of the frustrum of a right circular cone, using the accompanying figure.

1. By similar triangles show that $h=\dfrac{a\ r'}{r-r'}$

2. Show that the altitude of the entire cone is $\dfrac{a\ r}{r-r'}$

3. Show that the volume of the entire cone is $\dfrac{a\ \pi\ r^3}{3(r-r')}$

4. Show that the volume of the small cone is $\dfrac{a\ \pi\ r'^3}{3(r-r')}$

5. Show that the volume of the frustrum is a $\dfrac{a}{3}\ \pi(r^2+\sqrt{rr'}+r'^2)$

6. Show that $\dfrac{a}{3}\ \pi(r^2+\sqrt{rr'}+r'^2)=\frac{1}{3}\ a\ (b+\sqrt{bb'}+b')$

14. Use the formula given in Problem 13 to find how many gallons the bucket described in Problem 12 will hold.

15. If possible, the student should measure objects in the form of frustrums and find their surfaces and volumes.

184. Volume of Sphere. The method of finding the volume of a sphere may be discovered by comparing the sphere to a cylinder in which the altitude and diameter of base are both equal to the diameter of the sphere. The cylinder should be hollow so that the sphere may be fitted into it as shown in

Figure 60.

Figure 60. Now if the part of the cylinder not occupied by the sphere is filled with water and the sphere carefully removed it will be found that the water occupies just one-third of the cylinder. The volume of the sphere is therefore two-thirds the volume of the cylinder. Calling the radius of the sphere r, the altitude of the cylinder is $2r$ and the area of the base πr^2. The volume of the cylinder is therefore $2r \times \pi r^2$, or $2\pi r^3$. But the volume of the sphere is $\frac{2}{3}$ the volume of the cylinder, therefore the volume of the sphere is equal to $\frac{2}{3} \times 2 \pi r^3$, or $\frac{4}{3} \pi r^3$.

The volume of a sphere is equal to $\frac{4}{3}$ π times the cube of the radius, or $V = \frac{4}{3} \pi r^3$.

185. Surface of Sphere. To discover the method of finding the surface of a sphere take a large wooden sphere and saw it through the center. The plane circular surfaces of the hemispheres are called *great circles.* Then compare the curved surface of the hemisphere to the surface of the great circle. This can be done by driving a small nail into the center of the plane surface, tying a cord to it and wrapping the cord around the nail until the great circle is covered. Figure 61. Now measure the amount of cord used. Next drive the nail into the center of the curved surface of the hemisphere and wind the cord so as to cover this surface and again measure the amount of cord used, Figure 61. If this is done carefully it will be found that it takes twice as much cord to cover the curved surface of the hemisphere as it does to cover the great circle. The entire surface of the sphere is therefore four times the area of a great circle. But the

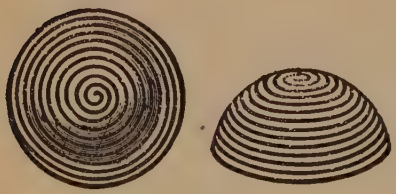


Figure 61.

radius of the great circle is the same as the radius of the sphere and the area of the great circle is πr^2, so it follows that

The surface of a sphere is equal to 4π times the square of its radius, or $S=4\pi r^2$.

186. Volumes of Irregular Shaped Solids. The volumes of small irregular shaped solids may sometimes be found by immersing the solid in water and measuring the volume of water displaced. One method of doing this is as follows. Partly fill a cylindrical vessel with water and measure the depth of the water. Immerse the object in the water and again measure the depth. The area of the base of the vessel multiplied by the increase in depth of the water will give the volume of the solid. Why?

Figure 62.

EXERCISES AND PROBLEMS. IRREGULAR BODIES. THE SPHERE.

1. If possible determine the volume of some irregular object by the method described.

2. Compare the curved surface of a hemisphere to the surface of its great circle by the method described.

3. Compare the volume of the sphere to the volume of the cylinder as described and develop the rule for the volume of the sphere.

4. In what ratio are the volumes of the following three figures (a) a sphere, (b) a cylinder having the diameter of its base and its altitude both equal to the diameter of the sphere, and (c) a cone having the same base and altitude as the cylinder? Draw a figure.

5. If possible, the student should measure some spherical object and determine its volume and surface.

6. A copper dome is in the form of a hemisphere 30′ in diameter. Find the weight of the copper in the dome if it weighs 1.5855 pounds per square foot.

7. A croquet ball is $3\frac{3}{8}''$ in diameter. If the ball is made from a block of hard maple $4\frac{1}{8}'' \times 3\frac{5}{8}'' \times 3\frac{5}{8}''$, how many cubic inches of maple are wasted in making a set of 8 balls? What per cent is wasted?

8. A dumb-bell is turned from a rectangular block of wood $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times 12''$. The two balls are each 3″ in diameter and the bar joining them is 4″ long and 1″ in diameter. How much will the dumb-bell weigh if the original piece weighed $5\frac{1}{2}$ pounds? Draw a figure.

9. A hollow cast-iron sphere is 1″ thick and 8″ in external diameter. How much does it weigh? (Cast-iron weighs .26 lb. per cubic inch).

10. A hay stack is in the approximate form of a cylinder surmounted by a hemisphere. It is 20′ in diameter and 18′ high. How many tons of hay does it contain? Draw a figure. (Allow 512 cu. ft. of hay to the ton).

11. Which weighs the most, two lead spheres each 1″ in diameter or one sphere of lead 2″ in diameter? How much more? Answer without finding the volume of any of the spheres.

VII. Field Work.

187. Need for Field Work. One of the best ways to make the study of measurement more interesting and more practical to the pupils is to give them opportunities to make simple measurements in the field. Problems such as measuring the width of a stream, the height of tower or smoke-stack, finding the area of the school grounds, making a map of the school yard, or of the village or part of the town in which they live, or laying off a tennis court or base ball diamond, are both interesting and profitable. All practical measurements in the field depend upon the measurement of quantities of two kinds, *distances* and *angles*.

188. Direct Measurement of Distances. One of the first field problems the pupils take should be the measurement of some distance, say the length of the school yard. The usual instrument used in measuring distances is the *tape line*. This should be either 50 ft. or 100 ft. in length and may be obtained divided into feet and inches, or into feet and tenths of a foot. The latter division is the most convenient and a tape line of this kind should be used if possible.

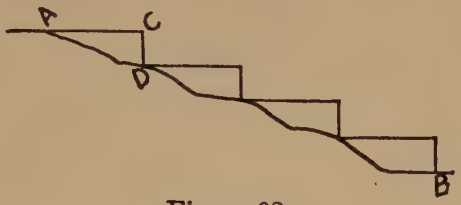


Figure 63.

In measuring the horizontal distance from A to B (Figure 63) hold the tape line taught and level. If the ground slopes the point D, immediately under the end of the tape C, can be determined by means of a plumb line. By repeating this process the horizontal distance from A to B may be found.

Figure 64.

If the two points are not connected by a straight line such as the edge of the school yard such a line should be marked out. This can be done for a short distance by stretching a string between the two points. To lay off a straight line between two points A and B, when the distance is too great to stretch a string, proceed as follows. Drive stakes at A and B. By sighting from A to B locate a third point C in the same straight line. Drive a stake at C. Then by starting at A and keeping B and C in line, as many points as desired can be located on the straight line between A to B and the line marked by means of stakes.

189. Direct Measurement of Angles. The instrument used in measuring angles is the *protractor,* a simple form of which is shown on page 165. Angles on drawings are easily measured by direct application of such a protractor.

The simplest instrument for measuring angles in field work is the *plane table.** "A convenient form of plane table consists of a drawing board about 24 by 28 inches mounted on three stakes which are three to four feet high after being driven into the ground. A small table or stand, blocked up to a convenient height, can be used for mounting the drawing board. Figure 65 represents a drawing board mounted as described and centered over a stake by means of a plummet."

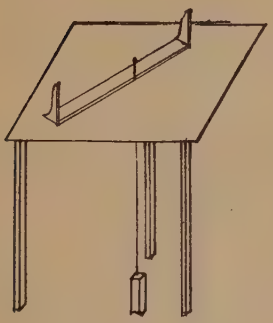


Figure 65.

"Figure 66 represents a sighter made of a straight edge ruler 18 or 24 inches long provided with a vertical sight at each end. The sights, which are about 4 inches high, have one edge perpendicular to the edge of the ruler and flush with it."

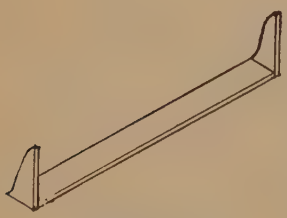


Figure 66.

To measure an angle, as the angle CAB formed by two intersecting roads (Figure 67), place the plane table so that a point A marked on the paper that has been tacked on the table shall be directly over the vertex of the angle to be measured, and stick a pin at A perpendicular to the table. Place the straight edge of the

*The description of the construction and use of a plane table given here is taken from "Plane and Solid Geometry," J. H. Williams and K. P. Williams.

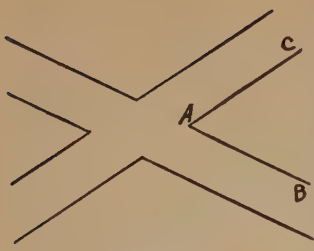


Figure 67.

sighter against the pin and swing it around until the straight edges of the sights are in line with the side AC of the angle. Hold the sighter firmly and mark line AC on the paper. Without disturbing the plane table, turn sighter until the sights are in line with line AB and mark line AB on the paper. The angle CAB on the paper can be measured by a protractor and thus the angle CAB between the roads can be determined. More accurate results will be obtained if the table is first leveled by means of a carpenters' level.

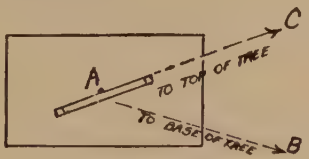


Figure 68.

An angle in a vertical plane can be measured by mounting the board in a vertical position as shown in Figure 68.

It is sometimes necessary in field work to lay off angles of a certain size. This can be done by first drawing the angle on a piece of paper by means of a protractor and then placing the angle on the plane table and sighting along the sides so as to locate points on the sides of the required angle in the field. Right angles can be laid off in this way or by means of a rope as described in Section 171.

190. Indirect Measurement. It is not always convenient or possible to measure the distance or angle desired directly. In such cases some indirect method must be used. The most

useful indirect methods for the elementary school are measurement by means of (1) the isosceles right triangle, (2) scale drawings, and (3) similar triangles.

191. Use of the Isosceles Right Triangle in Indirect Measurements. The only instruments needed are an isosceles right triangle, such as shown in Figure 69, made of heavy cardboard or light wood, and a stick or pole about 5 feet long. The use of the isosceles right triangle in measuring heights and distances that cannot be measured directly is illustrated in the following problems.

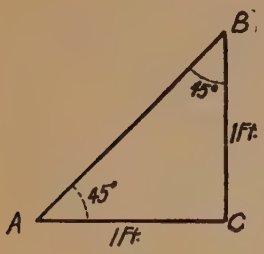


Figure 69.

Problem 1. To determine the height of a smoke stack such as shown in Figure 70.

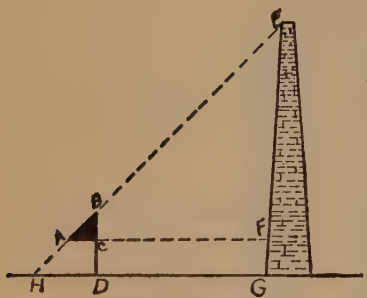


Figure 70.

Solution. Fasten one edge of the triangle ABC to the pole BD as shown in Figure 70. Holding the pole in a vertical position back away from the smoke stack until a position is found from which the top of the tower can just be seen when sighting along the the hypotenuse AB of the triangle. Measure the distance DG from the pole to the foot of the stack. This distance plus 1 foot (the length of AC) equals the distance AF. But AF=FE as the triangle AFE is isosceles. The height of the tower is this distance plus FG or CD which is known or can be measured. Or, after the pole is located, sight along the hypotenuse from B to A and locate the point H on the ground. Measure HG which equals the height of the stack.

Problem 2. To determine the distance between two points on opposite sides of a stream as D and E in Figure 71.

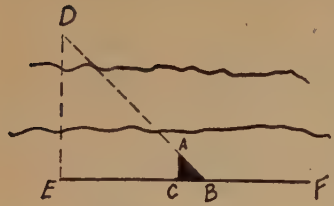


Figure 71.

Solution. Fasten the triangle ABC on top of the pole in a horizontal position. Lay off the line EF perpendicular to DE. Then, holding the pole in a vertical position and the side BC of the triangle ABC in line with EF, back along EF from E until a point is found so that D is in line with the hypotenuse AB of the triangle. Measure BE which is equal to ED.

192. The Use of Scale Drawings in Indirect Measurements. The use of scale drawings to determine distances and angles that cannot be directly measured is illustrated in the following problems.

Problem 1. To determine the distance between two points on opposite sides of the school building, as A and B in Figure 72.

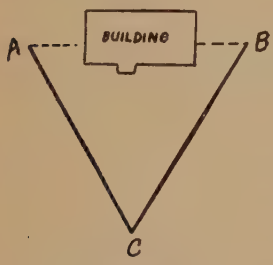


Figure 72.

Solution. Find some point C such that A and B can both be seen from C and that CA and CB can both be measured. Measure CA, CB and angle C. Draw on a sheet of paper an angle equal to angle C and choosing any convenient scale lay off on the sides of this angle distances to represent the lengths of CA and CB. Thus if CA=70 ft. and CB=85 ft. one inch on the drawing might represent 10 feet. Then CA would be drawn 7 inches long and CB would be drawn 8½ inches. Then draw AB and measure. If AB is found to be 9½ iches on the drawing and each inch represents 10 feet then the distance AB is 9½×10 ft.=95 ft.

Problem 2. To determine the angle between two lines when a plane table is not available.

Solution. Measure any convenient distance along the side BA of the angle to be measured, say 10 feet from B to D. Measure any convenient distance, say 10 feet, along the other side from B to E. Measure DE. Make a scale drawing of the triangle BDE and measure angle B with a protractor.

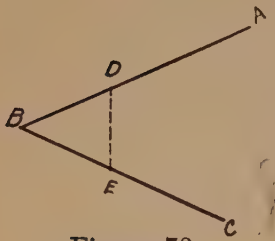


Figure 73.

193. The Use of Similar Triangles in Indirect Measurement. The fact that corresponding sides of similar triangles are proportional can often be made use of in indirect measurement as illustrated in the following problems.

Problem 1. To determine the height of a tree by the "Shadow Method."

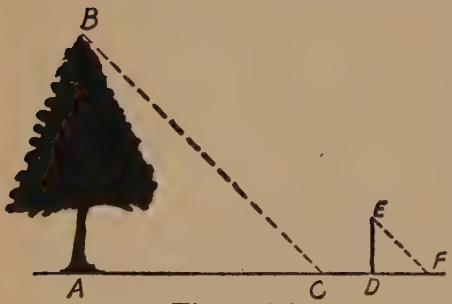


Figure 74.

Solution. Choose a time when the sun is shining and measure the length of the shadow AC of the tree. (Figure 74.) Hold a pole ED of known length, say 10 feet, in a vertical position and measure its shadow DF. Then since the triangles ABC and DEF are similar (they are right triangles and BC and EF are parallel rays of the sun so angle C=angle F) the sides are proportional and $\frac{AB}{ED}=\frac{AC}{DF}$, or $AB=\frac{AC\times ED}{DF}$. As AC, ED and DF are all known the height of the tree AB can be found.

Problem 2. To measure the length of a pond as AB in Figure 75.

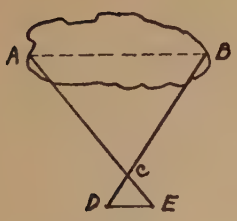


Figure 75.

Solution. Measure BC and AC. Make DC just ¼ as long as BC and CE just ¼ as long as AC. Then measure DE. AB is just 4 times DE as the two triangles are similar (two sides of one proportional to the two sides of the other and the included angles equal) and therefore their sides are in proportion, or $\frac{AB}{DE}=\frac{BC}{CD}=\frac{4}{1}$.

The size of an object may be determined from a photograph. During the war the positions of the enemy's guns, etc., were obtained by photographs taken from aeroplanes. Two things must be known, (1) the distance of the camera from the object when the picture was taken, and (2) the distance from the center of the lens of the camera to the plate or film. In the figure, triangles ALB and A′LB′ are similar and $\frac{AB}{A'B'}=\frac{CL}{LC'}$. A′B′ can be measured on the photograph, CL is the distance of the camera from the tree and LC′ is the distance from the lens to the film, so AB can be determined.

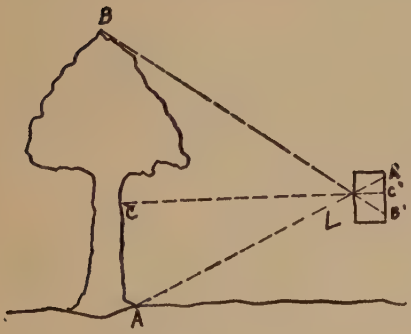


Figure 76.

EXERCISE. FIELD WORK.

1. Measure carefully the length and width of your school yard.

2. Measure carefully the distance between two points where there is no line between them to guide you.

3. If possible measure the horizontal distance between two points on sloping ground.

4. Construct an isosceles right triangle such as described in Section 191 and use it to measure the heights of various objects in the vicinity.

5. Stand at any point (A) in your school yard and look towards any prominent object (B). Turn and look toward another object (C). Measure the angle BAC by the method given in Problem 2 of Section 192. If possible also measure the same angle using a plane table.

6. A man walks along a road 10 miles due north until he comes to a road running southeast. He follows this second road for a distance of 8 miles, and then decides to return home across the fields. Make a scale drawing and find how far in a straight line the man is from his starting point.

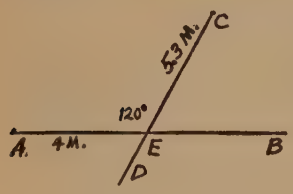


Figure 77.

7. The lines AB and CD in the figure represent two roads. By means of a scale drawing find what distance can be saved in walking from A to C by going across country in a straight line instead of following the roads AEC.

8. Take any two points A and B on the school grounds and without measuring it directly determine the distance AB by the method of Problem 1, Section 192. As a check, measure the distance AB.

9. By the same method determine the distance between two points on opposite sides of the school building.

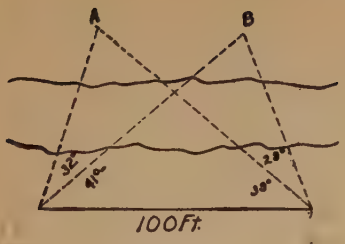


Figure 78.

10. To find the distance between two points A and B on the opposite side of a river, two boys laid out a line CD, 100 ft. long on the bank of the river and measured the two angles at C and the two angles at D. Draw to scale and find the distance AB. (The figure given is not a scale drawing).

11. Use the method of the preceding exercise to find the distance between two objects that can both be seen from the school grounds.

12. By use of the shadow method described in Problem 1 of Section 193, find the heights of several trees, telephone poles, smoke stacks, etc.

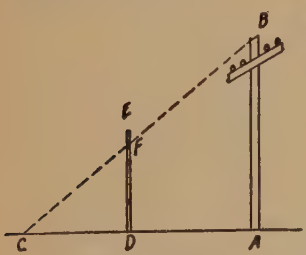


Figure 79.

13. Another method of finding the height of an object such as a telephone pole is shown in the accompanying figure. A long pole ED is fixed in the ground in a vertical position and at a convenient distance from the pole AB to be measured. One observer lies on the ground with his feet against the pole ED and his eyes at C. An assistant marks the point F on the pole ED which, to the eye of the observer, is in line with the top of the pole AB. The distances CD, CA and FD are measured. If CD=5′ 6″, CA=38′ and FD=8′ 9″, find AB.

14. Use the method of the preceding exercise to again determine the heights of the objects previously measured by the shadow method. How do the results compare?

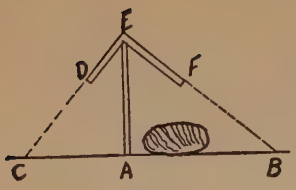


Figure 80.

15. The accompanying figure illustrates an old method of determining a distance AB which can not be directly measured because of an obstruction. DEF is an instrument similar to a carpenter's square. What measures would you take and how would you proceed to find the distance AB?

16. Use the method of Problem 2, Section 193 to measure the distance between two points on opposite sides of some obstruction such as the school building.

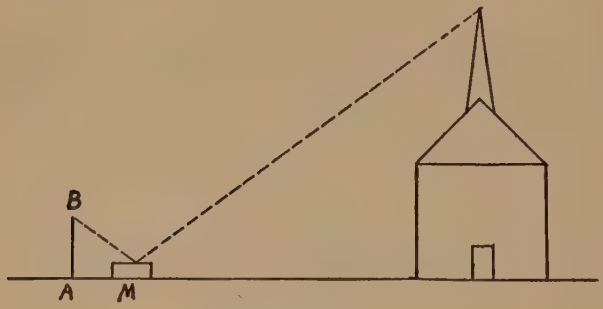


Figure 81.

17. The accompanying figure, Figure 81, shows a rough method of determining heights. Place a mirror on the ground at M and stand in a position AB from which the top of the church spire is seen in the center of the mirror. Tell how to determine the height of the spire. Use this method to find the height of some object.

18. If possible, determine the size of various objects in the vicinity of the school by photographing them and using the method described in Section 193.

CHAPTER XII.

GRAPHICAL REPRESENTATION OF STATISTICS.

EDUCATIONAL MEASUREMENTS.

I. Graphical Representation.

194. The Bar Graph. In order to show clearly the comparative sizes or values of various quantities, it is customary to represent the quantities by heavy lines or bars drawn to scale. Such a drawing is called a *bar graph*.

Figure 82 is a bar graph showing the number of acres in the United States planted to each of the principal grain crops in the year 1920. There were 104,000,000 (approx.) acres in corn, 54,000,000 (approx.) acres in wheat, etc.

Paper ruled in small squares, called cross-section paper, is convenient for use in constructing bar diagrams.

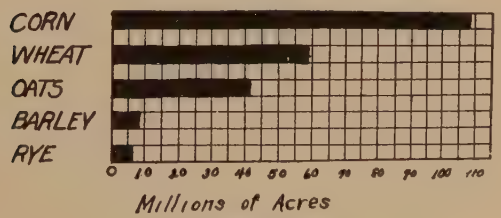


Figure 82.

195. Vertical Bar Graphs. In the previous section the bar graphs were drawn with the bars in a horizontal direction. It is also quite common to draw graphs with the bars in a vertical position. Figure 83 shows a graph of this kind. It

shows the number of pupils in a class in plane geometry who received grades below 60, from 60 to 64 inclusive, from 65 to 69, etc.

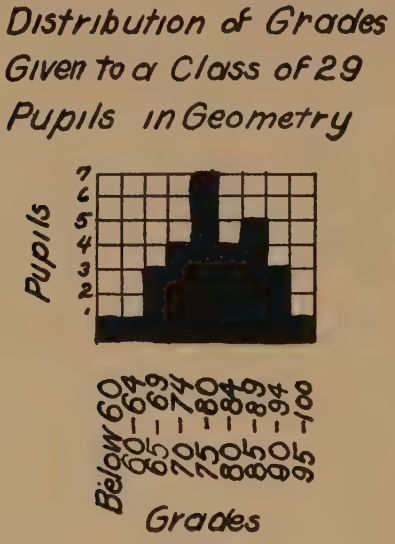


Figure 83.

196. The Circular Graph. It is often desired to show the relative size or value of the various parts of some whole as compared to each other and to the whole. A *circular graph* such as shown in Figure 84 is frequently used for this purpose. This graph shows the distribution of the industrial population in the United States among the different classes of industries in the year 1910. Of a total number of 38,167,336 persons engaged in industrial occupations 33.2% were engaged in agriculture, 27.9% in manufacturing, etc.

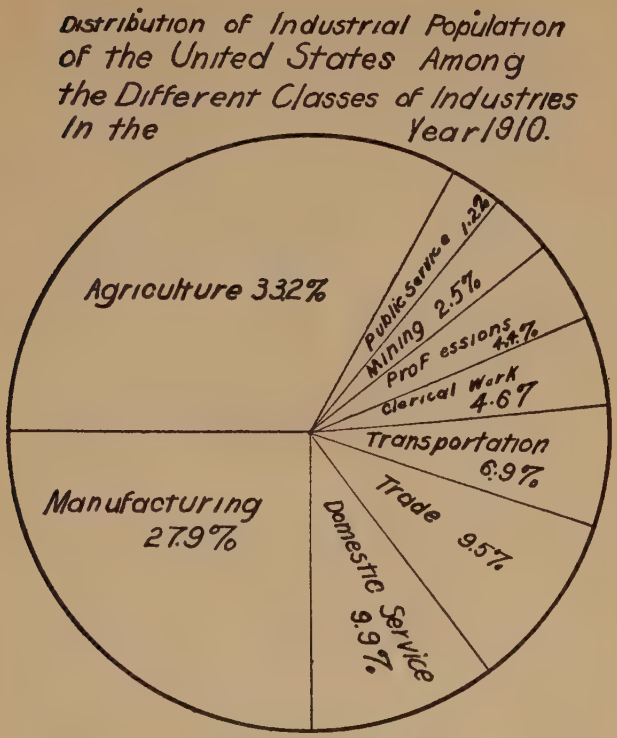


Figure 84.

197. The Line Graph. If it is desired to show the relationship between two quantities and how the one varies as the other changes, the line graph is usually used. Cross-section paper should be used for constructing such graphs. Figure 85 shows the increase in population in the United States from 1790 to 1920. The population in 1790 was 3,900,000 (approx.), in 1800 it was 5,300,000 (approx.), etc.

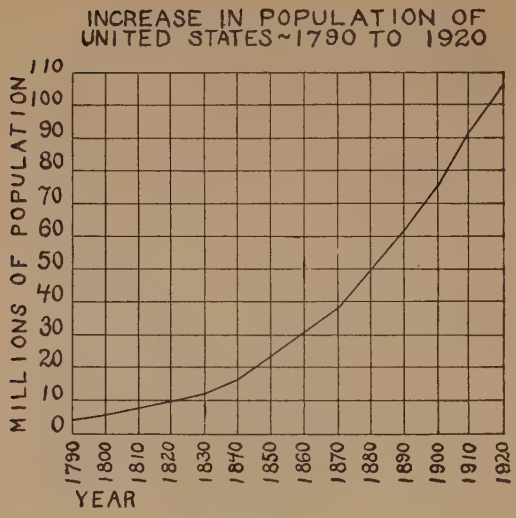


Figure 85.

EXERCISE. GRAPHS.

1. Find as many graphs as possible in current newspapers and magazines, government reports, books, etc. Bring these to class and be ready to explain them to the other students.

2. Construct a horizontal bar graph for each of the following sets of statistics.

a. Number of People Speaking the Principal Languages of the World.

English	German	Russian	French	Spanish
150,000,000	120,000,000	90,000,000	60,000,000	55,000,000

Italian	Portuguese
40,000,000	30,000,000

b. Population of Six Largest Cities of the United States.

New York	Chicago	Brooklyn	Philadelphia	Detroit	Cleveland
5,620,000	2,702,000	2,018,000	1,824,000	994,000	797,000

c. Number Head of Cattle in Various Countries.

India	United States	Russia	Brazil	Argentina
130,000,000	66,000,000	42,000,000	37,500,000	35,000,000

3. Make vertical bar graphs for the following.

a. Size of Farms in United States in 1920.

Under 20 acres	20-49 acres	50-99 acres	100-174 acres	175-499 acres	500 acres and over
800,000	1,500,000	1,480,000	1,450,000	1,010,000	220,000

b. Showing the per cent of A's, B's, C's, etc., given by each teacher in Problem 15, page 148.

c. Showing the standard distribution of grades given in Problem 14, page 148. Compare this with the two graphs in b.

4. Make a circular graph showing:

a. What per cent of the total number in your class are men and what per cent are women.

b. The per cent of the total number of students enrolled in your school who are in each year.

c. The distribution of total income suggested in Section 236.

d. What per cent of the total population of the United States in 1920 was native-born white, foreign-born white, negro, etc.

Total Population	Native-born white	Foreign-born white	Indian, Chinese, Japanese, and all other
105,710,620	81,108,161	13,712,754	426,574

e. The per cent of the families in Ohio that live in rented homes, in homes partly paid for, and in homes entirely paid for. For data see Problem 30, page 150.

5. Make line graphs showing:

a. The attendance record of your class for one month.

b. The increase in the population of Chicago from 1860 to 1920. For data see Problem 18, page 148.

c. The decrease in the *per cent* of the population of the United States living in rural communities from 1900 to 1920. For data see Problem 29, page 150.

d. Number of Automobiles in Use in the United States. 1915-1921.

1915	1916	1917	1918	1919	1920	1921
1,754,570	2,423,788	3,544,952	4,941,276	5,945,442	7,904,271	8,887,572

e. Increase in Railway Mileage of United States. 1850-1920.

1850	1860	1880	1900	1910	1920
8,588	30,626	93,267	193,345	240,438	253,152

f. Increase in Average Day Wages Paid Harvest Hands in the United States. 1894-1920.

1894	1895	1898	1899	1902	1910	1911	1912
$1.13	$1.14	$1.30	$1.37	$1.53	$1.82	$1.85	$1.87

1913	1914	1915	1916	1917	1918	1919	1920
$1.94	$1.91	$1.92	$2.07	$2.54	$3.22	$3.83	$4.36

6. On the same paper make line graphs showing the per cent of A's, B's, C's, etc., given by each teacher in Problem 15, page 148 and a third graph showing the standard distribution given in Problem 14, page 148.

II. Educational Measurements.

198. Measures of a Group. In connection with the standard tests in arithmetic and other school subjects it is necessary to obtain from the scores made by the individual pupils a score that represents the class as a whole. The two scores commonly used as a measure of the class are the *average* and the *median.*

199. The Average. The average of any set of measures is the sum of the measures divided by their number. A class of 15 eighth grade pupils made the following scores in examples attempted on the Courtis Standard Research Test in Addition; 6, 7, 9, 9, 9, 10, 11, 12, 12, 12, 14, 16, 17, 19, 21. The total of these fifteen scores is 186. The average is therefore $186 \div 15 = 12.4$.

200. The Median. The median of a group of measures is the middle measure. It is the measure above and below which there are equal numbers of the measures. Thus if the measures are 3, 3, 4, 5, 6, 6, 8, the median is 5 as there are three measures above it and three below. When a student makes a score of 6 examples attempted it does not necessarily mean that he tried just exactly 6 but means that he completed 6 but did not complete 7. A score of 6 then really means a score some place between 6 and 7, a score of 7 means one between 7 and 8, etc. In figuring medians it is convenient to arrange the measures or scores in order from the lowest to the highest putting opposite each score the number of times it occurs, as in the following table. Such a table is called a *frequency table.*

Distribution According to Speed of the Scores Made by a Class on an Addition Test.

Number of Examples Attempted	*Number of Times Score Occurred*
6- 7	1
7- 8	1
9-10	3
10-11	1
11-12	1
12-13	3
14-15	1
16-17	1
17-18	1
19-20	1
21-22	1
	15

As there are 15 scores altogether the median or middle score is the 8th score counting from the lowest. As there are 7 scores less than 12 the eighth or median score is evidently in the group of scores between 12 and 13. The middle point of this interval 12 to 13, or 12.5 may be taken as the *approximate median.* That is in general:

When the median score falls somewhere in a score interval the middle point of the interval is the approximate median.

In the above illustration if the score between 21 and 22 were missing so that there would be only 14 scores the median would then lie between the seventh score and the eighth score counting from the lowest. As the seventh score is somewhere between 11 and 12 and the eighth is between 12 and 13, the *approximate median* is taken as 12. Or in general.

When the median falls between two score intervals the upper limit of the lower interval or the lower limit of the upper interval is the approximate median.

201. Finding the True Median. The approximate methods of finding the median given in the preceding section are sufficiently accurate for most purposes. More reliable values for the median may in some cases be obtained by the method given below. A median obtained by this method is called a *true median.*

In the illustration first used above there are 15 scores and the median score is, therefore, the eighth counting from the lowest. The median score is therefore in the group of scores between 12 and 13. The middle point of this interval was taken as the approximate median. It is not likely, however, that the three scores between 12 and 13 are all exactly equal to $12\frac{1}{2}$, it is more probable that these scores are evenly distributed over the interval from 12 to 13, that is that one is between 12 and $12\frac{1}{3}$, another between $12\frac{1}{3}$ and $12\frac{2}{3}$, and the last between $12\frac{2}{3}$ and 13. Since the median score of the entire group is the eighth score and there are seven scores below 12, the eighth or median score is evidently the lowest of the three scores between 12 and 13 or the one lying between 12 and $12\frac{1}{3}$. The middle point of this interval 12 to $12\frac{1}{3}$, or $12\frac{1}{6}$ is therefore taken as the *true median.* The easiest way to get this is to think of the interval between 12 and 13 as being divided into three equal parts (3 scores) and the median is at the middle of the first of these parts or thirds. Therefore, the true median is $12+\frac{1}{2}\times\frac{1}{3}=12\frac{1}{6}=12.2$ approximately.

If there were an additional score between 20 and 21 in the above distribution, there would be altgether 16 scores and the median would be between the 8th and 9th score. But the 8th and 9th scores both lie in the interval 12 to 13. The approximate median is therefore the middle of this interval or 12.5. To find the true median we consider the three scores between 12 and 13 as being evenly distributed over the interval. That is the lowest is between 12 and $12\frac{1}{3}$, the next between $12\frac{1}{3}$ and $12\frac{2}{3}$, and the last between $12\frac{2}{3}$ and 13. As there are 7 scores below 12 and the

median of the group lies between the 8th and 9th scores, the median is evidently between the score in the interval 12 to $12\frac{1}{3}$ and the score in the interval $12\frac{1}{3}$ to $12\frac{2}{3}$. The true median is therefore $12\frac{1}{3}=12.3$ approximately. The easiest way to obtain this result is to think of the interval between 12 and 13 being divided into three equal parts (3 scores) and that the median lies between the first and second of these parts or at the end of the first part or third. The true median is then $12+\frac{1}{3}=12\frac{1}{3}=12.3$.

If the median is the third score in a group of 7 scores all in the same interval, say from 20 to 30, we think of the interval (10) as being divided into 7 equal parts (7 scores) and the median as being the midpoint of the third part. Then the true median is $20+2\frac{1}{2}\times\frac{1}{7}=20+\frac{5}{2}\times\frac{1}{7}=20+\frac{2}{7}=23\frac{4}{7}=23.6$ approximately.

If the median lies between the eighth and ninth scores in a group of 13 scores all in the same interval, say 10 to 20, we think of the interval as being divided into thirteenths and the median as lying between the eighth and ninth of these thirteen equal parts, that is at the end of the eighth thirteenth. The true median is then $10+8\times\frac{1}{13}=10+\frac{8}{13}=16\frac{2}{13}=16.2$ approximately.

202. Comparison of Average and Median. The median is usually taken as the measure of the class rather than the average, because it is easier to figure and is more typical of the group as a whole, as it is less affected by extremely good or bad scores. Suppose that in a class of seven pupils the individual scores of the pupils are 5, 5, 6, 7, 8, 8, and 20, respectively. The median (7) is more typical of the class than the average ($8\frac{3}{7}$) as the latter is unduly influenced by the one exceptionally good score of 20.

Problems. Educational Measurements.

1. Find the average, approximate median and true median in each of the following distributions.

(a)		(b)		(c)	
Score	Number of Children Making Score	Score	Number of Children Making Score	Score	Number of Children Making Score
1	..	1	..	1	1
2	..	2	..	2	7
3	2	3	4	3	7
4	7	4	16	4	23
5	6	5	10	5	10
6	14	6	15	6	14
7	6	7	13	7	2
8	3	8	13	8	4
9	3	9	7	..	..
10	..	10	2	..	..
11	..	11	1	..	..
12	1	..	..	..	..

(d)		(e)		(f)	
Score	Number of Children Making Score	Score	Number of Children Making Score	Score	Number of Children Making Score
0	1	0- 9	1	0- 4	..
1	..	10-19	3	5- 9	1
2	3	20-29	7	10-14	..
3	8	30-39	13	15-19	1
4	19	40-49	21	20-24	3
5	28	50-59	18	25-29	8
6	30	60-69	11	30-34	13
7	20	70-79	3	35-39	7
8	21	80-89	..	40-44	5
9	5	90-99	1	45-49	1
10	..	....	..	....	..
11	3	....	..	....	..
12	1	....	..	....	..
13	1	....	..	....	..
14	..	....	..	....	..
15	1	....	..	....	..

2. Make a vertical bar graph corresponding to each of the distributions given in 1.

3. A class of 33 pupils in the sixth grade made the following scores on the Courtis Test in Multiplication. The first figure given in each case is the number correct and the second figure is the number attempted.

Scores: 10-12, 12-15, 9-9, 5-8, 12-12, 11-14, 8-9, 8-10, 6-7, 10-11, 12-12, 14-15, 11-11, 11-12, 7-7, 4-5, 8-9, 10-11, 9-11, 8-8, 8-10, 10-10, 9-10, 11-12, 6-7, 9-9, 9-11, 11-11, 10-11, 9-9, 7-9, 10-10, 8-9.

a. Construct a frequency table showing the distribution of the class according to the number of examples attempted.

b. Compute the average, approximate median, and true median of examples attempted.

c. Figure the per cent of accuracy of each pupil. Carry the computation only far enough to determine in which of the following intervals the per cent lies, 0 to 49%, 50 to 59%, 60 to 69%, 70 to 79%, 80 to 89%, 90 to 99%, 100%.

d. Construct a frequency table showing the number of pupils, that have per cents of accuracy within each of the intervals given in (c).

e. Using the frequency table described in (d), find the approximate and true median per cents of accuracy.

f. Construct vertical bar graphs corresponding to the frequency tables described in (a) and (d).

4. Handle the following in the same way as suggested in 3. The scores given are those of an eighth grade class on the Courtis Test in Addition, given in September. The figures mean the same as in the preceding problem. Scores: 10-10,

2-12, 4-6, 1-4, 3-7, 4-8, 9-14, 7-12, 10-11, 5-7, 0-5, 1-6, 3-8, 3-9, 5-9, 7-10, 1-10, 6-16, 6-10, 2-6, 5-8, 5-10, 6-9, 5-9, 6-8, 3-8, 3-7, 3-5, 9-12, 8-10, 8-9, 6-7, 9-13, 5-10, 4-9, 10-11, 2-8, 2-10, 6-12, 5-9, 4-8, 3-8, 9-12, 6-10, 2-7.

5. If possible the class should take one or more standard arithmetic tests and handle their own scores in the way suggested above.

CHAPTER XIII.

ALGEBRA.

I. Algebra in the Elementary School.

203. Purpose. The chief purpose of the mathematics of the elementary school is to provide the pupils with a knowledge of the more important mathematical tools and with some degree of facility in the use of these tools in simple life situations. Arithmetic alone is not sufficient to accomplish this purpose. The mathematics of the shop and mechanical trades, and to a lesser degree that of the farm, the home and business involve not only arithmetic but simple algebra and geometry as well. As many of the boys and girls go into these occupations without a high school education it is necessary to include the more useful and elementary portions of algebra and geometry in the mathematics of the upper grades.

204. Extent. Only the most elementary portions of algebra are needed in the simple practical situations of life. Probably the one most important thing from the practical point of view is the ability to interpret and evaluate formulas and to change given formulas into others. Besides the work with formulas the algebra of the elementary school should include the simple equation, negative numbers, and the four processes with positive and negative numbers and with simple agebraic quantities.

II. The Formula.

205. Making Formulas from Rules. The formula should be introduced as an abbreviated method of writing the rules of arithmetic and the pupils should be given plenty of practice in making formulas from rules and in stating formulas as rules. The rules of mensuration afford excellent opportunity

for such practice. Thus the pupils discover that, *The area of a rectangle is equal to the product of the base and the altitude.* The possibility of writing this rule in an abbreviated form is next considered. The pupils are already familiar with such abbreviations as *N. Y.* for *New York, S. D.* for *South Dakota, in.* for *inch, P. S.* for *postscript,* etc. If these are reviewed they will have no difficulty in seeing that in a similar way we can use the abbreviations *A* for *area of a square, b* for *base* and *a* for *altitude.* The rule may then be written in the abbreviated form. *A is equal to the product of b and a.* Next they see that it can be still further abbreviated by the use of the familiar mathematical symbols $=$ for *is equal to* and $\times$ for *the product of.* The rule can then be written $A=b\times a$.

Each rule as it is discovered should first be stated in words and then in the abbreviated form of an algebraic formula.

206. Symbols. The symbols $+$, $-$, $\times$ and $\div$ are used in stating formulas as elsewhere in arithmetic. Multiplication is also sometimes indicated by a dot as $a\cdot b$, or by writing the two factors side by side without any sign between them as ab. Thus the product of a and b may be written $a\times b$, $a\cdot b$ or simply ab. Division is usually indicated by writing the dividend over the divisor in the form of a fraction as $\frac{a}{b}$ for $a\div b$.

207. Stating Formulas as Rules. In order to be able to use intelligently formulas that they have not made for themselves but have taken from some book, the pupils must be able to interpret the meaning of formulas. In order to be able to do this they must be given plenty of practice in taking formulas and restating them in words as rules. Thus the formula $T=N\times C$ in which T represents the *total cost* of a number of articles, N the *number* of articles, and C the *cost of each article,* should be translated into words something as follows:

The total cost of several articles is equal to the number of articles times the cost of each article.

After the rules have been stated as formulas the pupils should constantly be required to restate the formulas in words to be sure that the meaning of the formula is clearly understood.

208. Evaluating Formulas. In order to use a formula one must be able to substitute particular values for the letters involved, to carry out the indicated processes, and to simplify the result. This is called *evaluating* the formula. Thus the formula $A = \pi r^2$ gives a general rule for finding the area of any circle. If we want to find the area of a particular circle say one whose radius is 5 inches, we must substitute 5 for r and $3\frac{1}{7}$ for π in the general formula. $A = \pi r^2 = 2\frac{2}{7} \times 25 = 55\frac{5}{7}$ $78\frac{4}{7}$ square inches.

209. Order of Operations. Sometimes in evaluating formulas pupils make mistakes by performing the indicated processes in the wrong order. Thus in the formula $A = \frac{1}{2} \times a \times (b + b')$, if $a = 4$, $b = 7$, and $b' = 9$, pupils will sometimes proceed as follows: $A = \frac{1}{2} \times 4 \times (7 + 9) = 2 \times (7 + 9) = 14 + 9 = 23$. To avoid such mistakes the pupils must be acquainted with the following conventions concerning the order of performing the fundamental operations.

1. In a series of operations the multiplications and divisions are to be performed first and the additions and subtractions last. Thus $2 \times 7 + 3 \times 4 = 14 + 12 = 26$.

2. A quantity enclosed in parentheses is to be used as a unit. This means that the operations indicated within the parenthesis must be performed first. Thus $\frac{1}{2} \times 4 \times (7 + 9) = \frac{1}{2} \times 4 \times 16 = 32$.

The teacher should understand that these conventions are simply the results of arbitrary agreement among mathematicians and that different conventions might have been adopted. All of our elementary symbolism as we use it today is, however, based upon the universally accepted conventions stated above.

Exercise. The Formula.

1. State each of the following in the form of a formula.

1. The square of a fraction equals the square of the numerator divided by the square of the denominator.
2. To divide a fraction by an integer divide the numerator by the integer.
3. To multiply a fraction by an integer divide the denominator by the integer.
4. Multiplying both the numerator and denominator of a fraction by the same number does not change the value of the fraction.
5. The distance covered in a given time by a body moving at a uniform rate is equal to the product of the rate and the time.
6. The reading on a Fahrenheit thermometer is 32° more than $\frac{9}{5}$ of the reading on a Centigrade thermometer.
7. The arithmetical average of two numbers is equal to the sum of the numbers divided by 2.

2. State the following in words.

1. $C=n\times c$, if C represents the total cost, c the cost of one article and n is the number of articles.
2. $A=\sqrt{s(s-a)\ (s-b)\ (s-c)}$ of A represents the area of a triangle, s the semi-perimeter, and a, b and c the three sides.
3. $V=\frac{1}{3}\ h\ (b+b'+\sqrt{bb'})$, if V represents the volume, h the height, and b and b' the areas of the two bases of the frustrum of a right circular cone.
4. $A=P+I$, if A represents the amount, P the principal and I the interest.
5. $L=\frac{1}{2}\ s(c+c')$, if L represents the lateral surface, s the slant height, and c and c' the circumferences of the two bases of the frustrum of a right circular cone.

6. $V=\frac{\pi d^3}{6}$, if V represents the volume and d the diameter of a sphere.

7. $\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}$.

3. Evaluate the following.

1. $A=\frac{1}{2}\ b\times h$, if $b=7''$, $h=5''$.
2. $A=\frac{1}{2}\ h\ (b+b')$, if $h=9''$, $b=10''$, $b'=7''$,
3. $A=\sqrt{s(s-a)\ (s-b)\ (s-c)}$, if $a=11''$, $b=9''$, $c=5''$.
4. $c=\sqrt{a^2+b^2}$, if $a=7''$, $b=8''$.
5. $L=\frac{1}{2}\ s\ (c+c')$, if $s=9.5''$, $c=6''$, $c'=8''$.

III. The Equation.

210. Definition. The most important thing in mathematics is the *equation*. An equation is a statement that two numerical expressions are equal in value. All of the formulas previously considered as $A=\frac{1}{2}\ b\times a$, $V=\frac{4}{3}\ \pi\ r^3$, etc., are equations. The ability to handle equations successfully is the most important ability in mathematics. Although the formulas previously studied are equations it is best to start the study of the equation as such with simpler cases involving only one letter or unknown, such as $n+7=9$. This equation states in algebraic language that if some unknown number n is added to 7 the result is 9, and asks what is the value of the unknown number. It is obvious that in this case the value of n must be 2. Every equation thus does two things, makes a statement and asks a question. The question always is *what is the value of the unknown number represented by the letter?* The process of finding the value of the unknown in an equation is called *solving the equation.*

211. The Axioms. The solution of simple equations involving one unknown depends upon the knowledge of four fundamental truths or principles which are usually called the *axioms* of equations. These principles are as follows:

1. The same number may be added to both sides of an equation.

2. The same number may be subtracted from both sides of an equation.

3. Both sides of an equation may be multiplied by the same number.

4. Both sides of an equation may be divided by the same number.

Pupils must know and thoroughly understand these principles. Probably the best way to show their truth is to think of the equation as a balance of values and compare it to a simple balance or pair of scales such as shown in Figure 86.

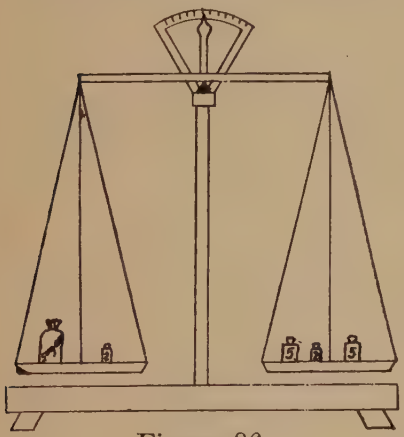


Figure 86.

Suppose a sack of sugar of unknown weight and a 2 lb. weight are placed on the left pan of the balance and it is found that it takes two weights of 5 lbs. each and a weight of 2 lbs. on the right pan to balance. Then the weight of the sugar plus 2 lbs. is evidently equal to 5 lbs. plus 5 lbs. plus 2 lbs., or if W represents the weight of the sugar, $W+2=5+5+2$, or $W+2=12$. Now if the 2 lb. weight is removed from the left pan the scales will not balance until the 2 lb. weight is removed from the right side also. When this is done the scales will again balance and it is evident that the weight of the sugar is equal to 12 lbs. minus 2 lbs., or $W=12-2$, or $W=10$. From this and similar illustrations it is easy to make it clear to the pupils that taking equal amounts from both pans of the scale will not destroy the balance and that subtracting

the same number from both sides of an equation will not destroy the equality of the two sides.

If 3 lbs. is placed in each pan of the scale it will evidently balance. Now if the amount on each side is multiplied by 4 there would be 12 lbs. on each side and the scale would still balance. Thus it is seen that multiplying both sides of an equation by the same number does not destroy the equality of the two sides. The truth of the other axioms can be illustrated in a similar fashion. This is left to the student as an exercise.

212. The Solution of Equations. To solve an equation means to find the value of the unknown or the letter involved in the equation. To find the value of the unknown the terms or numbers involved in the equation must be rearranged so that the unknown number is on one side of the equation by itself and all the other numbers are on the other. This rearranging is done by means of the axioms. Thus in the equation $n+3=8$ we arrange so that n is on the left by itself and the other numbers are all on the right side of the equation. To start with 3 is on the left side, so to get rid of it we subtract 3 from both sides of the equation giving $n=8-3$, or $n=5$.

In the equation $n-2=7$, we want to change the equation so that we will have n on the left side instead of $n-2$. Now $n-2$ is 2 less than n so we must add 2 to $n-2$ to change it to n. Therefore, we must add 2 to both sides of the equation. Thus $n-2=7$, adding 2 to both sides we get $n=7+2$, or $n=9$.

In the equation $3\,n=12$, to get n on the left side, we must divide both sides of the equation by 3 as $3n$ is 3 times n. Doing this the equation becomes $n=4$.

In the equation $\frac{1}{2}\,n=7$, to get n on the left side both sides must be multiplied by 2 as n is just 2 times as large as $\frac{1}{2}\,n$ Doing this the equation becomes $n=14$.

213. Checking the Solution. After the value of the unknown has been found the result should be proven or checked to make sure that no mistake has been made. The best way to

do this is to put the value obtained for the unknown in place of the unknown in the original equation to see if it *satisfies* the equation, that is makes both sides equal. Thus in solving the equation $n-2=7$, above we found $n=9$. To check this result we put 9 in the place of n in the original equation $n-2=7$ and notice whether both sides are equal. Doing this we get $9-2=7$, or $7=7$, so it is evident that 9 is the correct value of n. The putting of 9, the value of n, in the place of n is called *substitution*.

Exercise. The Axioms.

1. By means of the balance show the truth of the addition and division axioms.

2. Find the value of the unknown in each of the following equations. Tell in each case just what you do to each side of the equation that is which axiom is used. Check the value obtained for the unknown by substitution.

1. $3x=15$	10. $\frac{1}{7}x=8$
2. $4x=24$	11. $3x+2=11$
3. $2x=9$	12. $2x-5=7$
4. $x+5=7$	13. $\frac{2}{3}x=8$
5. $x+3=8$	14. $\frac{3}{4}x+1=7$
6. $x-4=9$	15. $\frac{3}{5}x-2=4$
7. $x-5=1$	16. $7x-3=9$
8. $x-7=3$	17. $3x+5=7$
9. $\frac{1}{3}x=9$	18. $\frac{5}{8}x-2=14$

214. Using Equations in Solving Problems. The problem in algebra serves two important purposes. In the first place it makes the formal work more interesting by providing a motive, by showing the pupils that algebra is a useful tool in solving practical problems. In the second place, if properly taught, it develops the ability to apply algebra to simple situations. It is through the work with written problems that the pupils learn to "size-up" a practical situation, to express the conditions of the

situation algebraically and finally to solve it. Facility in the formal phases of algebra is certainly important, but the ability to apply algebra to concrete situations is at least equally so.

Any problem involves certain quantities, known and unknown. If the problem is capable of solution enough facts about these quantities and the relations existing between them must be known or given in the problem to enable the pupil to do two things, (1) to express each of the unknowns algebraically in terms of some one letter, and (2) to write an algebraic equation or sentence stating a relation between some or all of the quantities.

The complete solution of a problem using one letter involves five steps:

I. The abstracting of the problem, or getting clearly in mind three things, (a) what unknown quantities are involved in the problem, (b) what is known or given about these quantities, and (c) what is to be found.

II. The forming of the vocabulary, or the algebraic representation of the unknown quantities involved in the problem. This will use all but one of the relations given or known.

III. The forming of the equation, or the stating of the remaining relationship in the form of an algebraic sentence or equation.

IV. The solution of the equation.

V. The checking of the solution by seeing if it satisfies the conditions of the problem (*not the equation*).

Of these five steps the second and the third are the most difficult and pupils must be given much practice on these as a preparation for and in connection with the solution of problems.

215. Representing the Quantities in a Problem by Means of One Letter. The ability to represent quantities by algebraic expressions can only be obtained through practice. A great deal of this practice must be given and should be of two kinds,

(a) the translating of related algebraic expressions into English and (b) the translating of English descriptions of related quantities into algebraic expressions in one letter. The following illustrations should make this clear. (a) If n represents one number what does $n+2$ represent? and (b) if n represents one number what will represent a number 3 times as large? The translating of related algebraic expressions into English is the easier of the two and should come first. Practice of both kinds should be given not only when problems are first introduced but should be kept up at intervals throughout the course.

216. The Formation of Equations. The ability to express relationships between quantities by means of algebraic equations is developed only through practice. This practice should again be of two kinds, (a) translating algebraic equations into English, and (b) translating English statements of relationships between quantities into algebraic equations. The following examples should make clear the character of this drill. (a) $\frac{1}{2}n=n-2$. This equation translated into English states that half of a certain number is 2 less than the number itself. (b) If 42 be added to a certain number the result is 7 times the original number. If the number is represented by n this statement can be translated into the algebraic equation $n+42=7n$.

The work in translating rules into formulas and formulas into rules, previously suggested, also gives excellent drill in expressing relationships by means of algebraic equations.

The first problems met by the pupils should involve only one unknown. The algebraic representation of the unknown in such problems presents no difficulty, the sole difficulty being that of translating the relation given in the problem into an equation.

217. General Directions for the Solution of Problems.

I. Read the problem carefully and make a list of the unknown quantities involved.

II. Represent one of these quantities by some letter, usually the initial letter of the name of the quantity. Then from the

statements of the problem find an algebraic expression to represent each of the other unknown quantities. This will make use of all but one of the relations stated in the problem.

III. Express the remaining relationship given in the problem in the form of an equation using the algebraic representations of the quantities involved.

IV. Solve the equation.

V. Check by seeing if the values found satisfy the conditions of the problem.

Problem 1. (One unknown quantity) I am thinking of a number. If I double it and subtract 7, I have 5 left. What is the number?

Solution.

1. Let n=the number I am thinking of
2. Then $2n-7=5$

Explanation. (1) The only unknown in the problem is the number I am thinking of so we represent it by n.

(2) The problem states that twice the number less 7 leaves 5. Stating this algebraically we have the equation $2n-7=5$.

Problem 2. I bought a tablet and pencil. The tablet cost 5¢ more than the pencil and both together cost 13¢. Find what I paid for each.

Solution.

1. Let c=no. cents paid for pencil.
   Then $c+5$=no. cents paid for tablet.

Explanation. (1) The problem involves two unknowns, the number of cents paid for the pencil and the number of cents paid for the tablet, so we represent one of these, the number of cents paid for the pencil, by c. Then the problem states that I paid 5¢ more for the tablet than for the pencil so $c+5$ represents the number of cents paid for the tablet.

2. Then
$$c+(c+5)=13$$

(2) We have now used all the relations stated in the problem except the fact that the cost of the pencil and the tablet both was 13¢. Expressing this as an algebraic equation we get $c+(c+5)=13$.

EXERCISE. ALGEBRAIC REPRESENTATION. PROBLEMS.

I. The following exercises give practice in translating related algebraic expressions into English.

1. If d equals the number of dollars paid for a sled what does $2d$ equal? $5d$?

2. If n represents one number what does $n+6$ represent? $n-3$? $\frac{n}{2}$? $\frac{2n}{3}$?

3. If x represents A's age in years today what does $x-3$ represent? $x+5$?

4. If y represents A's age ten years ago what does $y+10$ represent? $y+6$? $y+12$?

5. If n represents a certain number what does $n+3$ represent? $n(n+3)$? $n+(n+3)$? $(n+3)-n$? $\frac{n}{n+3}$? $\frac{n+3}{n}$?

II. The following exercises give practice in translating English descriptions of related quantities into algebraic expressions in one letter.

1. If n represents a certain number what will represent a number 5 times as large? Half as large? Three-fourths as large? Seven less? Nine more?

2. If y represents the number of years in John's age what will represent the number of years in his father's age if John's father is just 30 years older than John? What will represent the age of John's older brother if he is 5 years older than

John? What will represent the age of his sister who is 2 years younger? Of his grandfather who is just twice as old as John's father? Of his grandmother who is 5 years younger than his grandfather?

3. If C is now y years old how old was he 7 years ago? How old will he be in 3 years?

4. If a train runs at the rate of 28 miles an hour how far does it run in 3 hours? In h hours?

5. If x represents the number of inches in the side of a square, what represents the number of square inches in the area?

6. How many cents are there in d dollars? In q quarters?

7. If a man earns \$7 a day how much will he earn in n days?

8. A rectangle is twice as long as it is wide. If w represents the number of inches in its width what represnts the length? The perimeter? The area?

9. A rectangle is 6 feet longer than it is wide. If w represents its width what represents its length, perimeter and area?

10. Think of any number and remember it, add 8, add the original number, divide by 2, and subtract the original number. Prove that the result will always be 4 no matter what number is taken to start with. (Hint. Represent the original number by n and then represent the result of each of the steps in terms of n.)

III. The following exercises give practice in translating algebraic equations into English statements.

1. Translate the following equations into English if in each n represents a certain number.

a. $n+3=7$
b. $n-5=9$
c. $2n=18$
d. $\frac{n}{2}=7$
e. $\frac{2}{3}n=5$
f. $2n-1=9$
g. $3n+7=19$
h. $7-n=2$

2. State the following equations in English if y equals the number of years in John's age and $y+30$=the number of years in his father's age.

a. $y+(y+30)=60$
b. $(y+30)=3y$
c. $y=\frac{1}{3}(y+30)$
d. $(y+30)+2y=75$
e. $2(y+30)-3y=45$

3. If L represents the number of feet in the length of a rectangle and L—7 the number of feet in the width, translate the following equations into English.

1. $L(L-7)=18$
2. $2L+2(L-7)=22$

IV. The following exercises give practice in translating English statements of relationships between quantities into algebraic equations.

1. Express the following statements as equations.

a. The sum of x and 3 is 5.

b. If 7 is subtracted from n the result is 4.

c. x is 7 less than 12. (Two ways).

d. y is 3 more than 8. (Two ways).

e. 32 is 4 times x.

f. If we take $\frac{2}{3}$ of n the result is 9.

g. If n is subtracted from 52 the result is the same as obtained by adding 18 to n.

h. Three times n is 7 more than 13. (Two ways).

i. Three times n is 18 less than 32 more than n. (Two ways).

2. If d represents the number of dollars A has and $d+7$ represents the number of dollars B has, express the following statements as equations .

a. A and B together have $17.

b. A has $\frac{5}{12}$ as much as B.

c. If B gives A twice as much as A has to start with B will have \$2 left.

d. If B gives A \$3.50 they will both have the same amount.

e. A and B together have $3\frac{2}{5}$ as much as A alone.

V. Solve the following problems. Check.

1. A girl spends \$15 for a hat and a pair of shoes. The shoes cost one and one-half times as much as the hat. Find the cost of each.

2. A furniture dealer who sells goods on installments allows a discount of 20% or $\frac{1}{5}$ from the marked price of all articles if cash is paid. At what must he mark a chair in order to receive a cash price of \$20?

3. Mary asked Jane to think of any number, add 6, multiply by 2, and subtract 5. Jane then told that the result was 11 and Mary said that Jane started with the number 2. How did Mary know this? (Hint. Let n represent the original number and represent the result of each step in terms of n. Set the final result equal to 11 and solve for n.)

4. A clothing dealer sold an overcoat at \$50. If he sold it at a gross gain of 25% or $\frac{1}{4}$ of the cost what did the overcoat cost him?

218. Obtaining New Formulas From Old. The axioms of equations can also be used in obtaining new formulas from old. Thus the formula $A = \pi r^2$ can be used in finding the area of a circle when the radius is known. If a formula is wanted to use in finding the radius when the area is known it can be obtained by solving the original formula for r or r^2. From the original formula $\pi r^2 = A$. Dividing both sides by π we get $r^2 = \frac{A}{\pi}$ and $r = \sqrt{\frac{A}{\pi}}$.

EXERCISE. OBTAINING NEW FORMULAS.

1. From the formula $A = \pi r^2$ derive a formula for finding the radius of a circle when the area is known.

2. From the formula $C = 2\pi r$ derive a formula for finding the radius of a circle when the circumference is known.

3. From the formula $A=b\times h$ derive a formula for finding the base when the area and altitude of a rectangle are known. One for finding the altitude.

4. From the formula $V=\frac{4}{3}\pi r^3$ derive a formula for finding the radius of a sphere when its volume is known.

5. The formula for changing readings of the Fahrenheit thermometer into those of the Centigrade is $C=\frac{5}{9}(F-32)$. From this derive a formula for finding the Fahrenheit reading when the Centigrade reading is known.

IV. Negative Numbers.

219. The Need of a New Kind of Number. So far in mathematics the pupil has dealt with quantities that are measured in only one direction from the zero or starting point. It must now be called to his attention that many quantities can be measured in two opposite directions form the starting point or zero. Some of the most familiar quantities of this kind are, (1) Temperatures above and below zero, (2) Altitudes above and below sea level, (3) North and south latitude, (4) East and west longitude, (5) Assets and debts, (6) Scores and penalties in games, (7) Distance in any direction and distances in the opposite direction, (8) Time B. C. and A. D.

In describing quantities of this kind it is not sufficient simply to give a number that shows the *magnitude* of the quantity, but it is also necessary to indicate in some way the *direction* of the quantity from the starting point or zero. Thus to say that the temperature on a certain day was 5° means nothing unless we know whether it was above or below zero. To say that the latitude of a certain place is 20° means little unless we know whether it is north or south.

In such cases it is customary to call quantities measured in one direction from the zero *positive* and to indicate them by means of the plus sign +, and those measured in the opposite direction *negative* and to indicate them by means of the minus

sign —. Thus +2, +7 and +3 are positive numbers and —3, —5, —8 are negative numbers.

The following table gives a list of quantities which it is customary to regard as positive. The student should state the corresponding negative quantity in each case.

Positive Quantities.

1. Temperatures above zero.
2. Altitudes above sea level.
3. North latitude.
4. East longitude.
5. Assets.
6. Distances to the right.
7. Distances up.
8. Time B. C.
9. Points made in a game.

Considerations such as those given above should make clear to the pupils the meaning of negative numbers and that negative numbers are just as real as positive and just as necessary in dealing with practical affairs.

220. The Absolute and Algebraic Values of Positive and Negative Numbers. To completely describe a quantity that may be measured in either of two opposite directions from zero, it is necessary to give the sign + or — to indicate the direction, and a number to indicate the distance from zero or the magnitude of the quantity. The number which indicates the magnitude of the quantity regardless of the sign is called the *absolute* value of the number. Thus the absolute value of both +7 and —7 is the same, namely 7. Both represent quantities of the same magnitude but of opposite kinds. The value of a number when the sign as well as the magnitude is considered is called the *algebraic* value. Thus +7 and —7 differ in algebraic value although their absolute values are the same.

221. The Addition of Positive and Negative Numbers. As soon as the meaning of negative numbers is understood the pupils must learn how to work with them, how to add, subtract, multiply and divide. The various illustrations of positive and negative numbers given above are not only useful in making clear the meaning of negative numbers but they can also be used to help the pupil discover how to work with such numbers.

Addition is a process of combining numbers. The pupils already know how to combine or add positive numbers. They must now discover how to combine negative numbers and positive and negative numbers. There are five different cases to be considered. (1) Both numbers positive, (2) both numbers negative, (3) the two numbers having the same absolute value but one positive and the other negative, (4) one positive and one negative, the positive number having the greater absolute value, and (5) one positive and one negative, the negative number having the greater absolute value. These five cases are illustrated by the following examples.

$$\begin{array}{ccccc} +3 & -4 & -7 & -7 & -8 \\ \underline{+5} & \underline{-2} & \underline{+7} & \underline{+9} & \underline{+5} \end{array}$$

Case I. Both numbers positive. This case really involves nothing new as all the numbers dealt with in arithmetic are positive numbers. It should be reviewed here, however, and included in the general summary. The pupils can take cases such as $\begin{array}{ccc} +3 & +7 & +6 \\ \underline{+5} & \underline{+4} & \underline{+8} \end{array}$ and reason them out by using any of the illustrations of positive and negative numbers given above. Thus if you have a score of $+3$ in some game and score again getting a score of $+5$ then your total score is $+8$. That is $+3$. Or if you start at any point and take three steps

$$\begin{array}{r} +5 \\ \hline +8 \end{array}$$

to the right (+3) and then take five steps to the right (+5) the two together make eight steps to the right (+8). This is illusrated on the diagram in Figure 87.

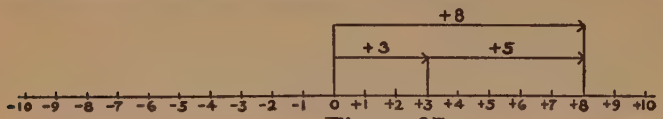


Figure 87.

Case II. Both numbers negative. If you have a score of 5 "in the hole" (—5) and play again and incur a penalty of 4 (a score of —4) your total score is 9 "in the hole" or —9. Or, if you take 5 steps to the left (—5) and then take 4 steps to the left (—4) the two together make 9 steps to the left of the starting point (—9). That is $\begin{array}{r} -5 \\ -4 \\ \hline -9 \end{array}$

Having found that $\begin{array}{r} +3 \\ +5 \\ \hline +8 \end{array}$ and that $\begin{array}{r} -5 \\ -4 \\ \hline -9 \end{array}$ and other similar facts the pupils can form the following generalization.

To add numbers having the same signs add their absolute values and give the result the same sign as the original numbers.

Case III. One number positive, the other negative, both having the same absolute value. If one has a score of +7 and plays again incurring a penalty of 7 (score of —7) his total score is 0. If you take 7 steps to the right (+7) and then 7 steps to the left (—7) the two together leave you where you started or at 0. That is $\begin{array}{r} +7 \\ -7 \\ \hline 0 \end{array}$ From cases such as these pupils should see that when you combine (add) positive and negative numbers you are combining numbers of opposite kinds or numbers measured in opposite directions and that any number of positive units combined with the same number of negative

units will give zero. Another excellent illustration of this is the "tug of war." If seven men are pulling on one end of a rope (+7) and seven men of equal strength are pulling on the other end (−7) the rope will not move, the 7 positive just neutralize or balance the 7 negatives and the result is zero.

Case IV. One number positive the other negative, the positive number having the greater absolute value. If you have a score of 9 in a game (+9) and incur a penalty of 5 (−5) your total score is +4. Or if you take 9 steps to the right (+9) and 5 steps to the left (−5) the two together leave you 4 steps to the right of the starting point (+4). That is

$$\begin{array}{r} +9 \\ -5 \\ \hline +4 \end{array}$$

Case V. One number positive, the other negative, the negative number having the greater absolute value. If you have a score of −8 (eight "in the hole") and play again scoring five (+5) your total score is −3 (three "in the hole"). Or if you take 8 steps to the left (−8) and then take five steps to the right (+5) the two together leave you three steps to the left of the starting point (−3). That is −8.

$$\begin{array}{r} +5 \\ \hline -3 \end{array}$$

From cases such as +9 and −8 and other similar cases

$$\begin{array}{r} -5 \\ \hline +4 \end{array} \qquad \begin{array}{r} +3 \\ \hline -5 \end{array}$$

the pupils should see that combining (adding) numbers having different signs does not combine their absolute values but as the numbers are of opposite kinds, are figuratively speaking pulling on opposite ends of the "tug of war," equal numbers of positive and negative units will just balance each other and the result will be in the direction of the number having the largest absolute value and will be equal in magnitude to the difference in absolute values of the original numbers. Thus,

when -2 is combined with $+8$, the 2 negatives will just balance or neutralize 2 of the positives and the result will be 6 positives. From considerations such as these the pupils can draw the following general conclusion.

To add numbers having different signs find the difference between their absolute values and give this result the sign of the number having the larger absolute value.

222. The Subtraction of Positive and Negative Numbers. There are several possible ways of subtracting positive and negative numbers. The method usually given in our texts, which is probably as satisfactory as any, is based upon the following principles.

1. Subtracting a positive number is the same as adding a negative number having the same absolute value.

2. Subtracting a negative number is the same as adding a positive number having the same absolute value.

The truth of these two principles may be illustrated in a variety of ways. The following illustrations show the truth of the first principle.

From $+8$ subtract $+2$. The pupils already know how to do this and get the result $+6$. To $+8$ add -2. Again the pupils know how to do this and get the result $+6$. They thus see that subtracting $+2$ from $+8$ gives the same result as adding -2 to a $+8$. Other similar illustrations may be used.

Another illustration of the principle may be based on scores in games. A penalty of 2 incurred in a game may be thought of in either of two ways, either as taking 2 from the previous score (subtracting $+2$), or as giving a new score of -2 (adding -2). From this it is seen that subtracting $+2$ means the same thing as adding -2.

A third illustration is based on steps taken in two opposite directions. If you have already taken 8 steps to the right ($+8$) and next take 2 steps to the left you can think of it in either of two ways, either as taking 2 new steps in a negative direction

(adding -2 steps to those already taken), or as undoing or taking away 2 of the steps previously taken (subtracting $+2$ steps from those previously taken).

This principle may also be illustrated by the "tug of war." If there are 8 men pulling on each end of a rope, the effect of taking 2 men from the positive end (subtracting $+2$) is the same as adding two more men to the negative end (adding -2).

The second principle may be illustrated in similar ways. If we have 9 negative units and take away 4 negative units there are evidently 5 negative units left, that is if -4 is subtracted from -9 the result is -5. But if we add $+4$ to -9 the result is also -5. From this it is seen that subtracting -4 from -9 gives the same result as adding $+4$ to -9.

If in some game you make a play and incur a penalty of 3 (a score of -3) and later the rules are consulted and it is found that the penalty should not have been imposed it is evident that your score can be corrected in either of two ways, either by taking off the penalty of 3 (subtracting -3) or by adding 3 more to your score (adding $+3$).

If you first take 9 steps to the left (-9) and then take 3 steps to the right you can think of it in either of two ways, either as taking 3 new steps in a positive direction (adding $+3$ steps to those previously taken), or as undoing or taking away 3 of the steps previously taken to the left (subtracting -3 steps from those previously taken).

This principle may also be illustrated by the "tug of war." This is left to the student as an exercise.

When the truth of the two principles given above is evident to the pupils they can see that since subtracting a positive number gives the same result as adding a negative and since subtracting a negative number gives the same result as adding a positive it follows that an easy way to subtract positive and negative numbers is:

To subtract one number from another change the sign of the subtrahend and add the result to the minuend.

In using this method of subtracting the change of signs should be made *mentally* and the *addition must be performed according to the rules for adding positive and negative numbers.*

223. The Multiplication of Positive and Negative Numbers. Four cases arise in the multiplication of positive and negative numbers (1) positive times positive, (2) positive times negative, (3) negative times positive, and (4) negative times negative.

The first two cases can be based directly on the meaning of multiplication by a positive number. Thus $3\times4=4+4+4=12$ and $3\times(-4)=(-4)+(-4)+(-4)=-12$.

The third case can be worked out by using the principle that in the case of abstract numbers the multiplier and multiplicand may be interchanged without changing the result. Thus $(-3)\times4=4\times(-3)=-12$. From this a *definition* of the meaning of multiplication by a negative number may be derived. Since $(-3)\times4$ is found to equal -12 and since 3×4 equals 12 we can form the following definition.

To multiply by a negative number means to multiply by its absolute value and change the sign of the product.

The fourth case can then be based on this definition. Thus $(-3)\times(-4)$ means to multiply -4 by 3 (the absolute value of the multiplier) which gives -12 and then to change the sign of this product which gives $+12$. So $(-3)\times(-4)=+12$.

The method used in deriving the laws of multiplication will depend upon the definition adopted for multiplication by a negative number. A second method of derivation based on a different definition follows. As before $3\times4=4+4+4=12$ and $3\times(-4)=(-4)+(-4)+(-4)=-12$. That is to multiply by a positive number means to add the multiplicand as many times as indicated by the multiplier, starting with zero. A reasonable definition to adopt for multiplication by a negative number would then be:

To multiply by a negative number means to subtract the multiplicand as many times as indicated by the multiplier, starting with zero.

The last two cases can be based on this definition. Thus $(-3)\times 4$ means to subtract 4 three times starting from zero. $0-4=-4$, $-4-4=-8$ and $-8-4=-12$, so $(-3)\times 4=-12$. Also $(-3)\times(-4)$ means to subtract -4 three times starting with zero. $0-(-4)=4$, $4-(-4)=8$, $8-(-4)=12$, so $(-3)\times(-4)=+12$.

From these four cases $3\times 4=12$, $3\times(-4)=-12$, $(-3)\times 4=-12$, and $(-3)\times(-4)=+12$, and other similar cases the pupils can form the general conclusion:

To multiply positive and negative numbers, multiply their absolute values. If the numbers have the same sign the product is positive, if they have different signs the product is negative.

224. Division of Positive and Negative Numbers. The division of positive and negative numbers is best based on the principle that the product of the quotient and the divisor always equals the dividend. Thus $12\div 3=4$ and the product of the quotient 3 and the divisor 4 equals the dividend 12. The same four cases arise in division as in multiplication. The first of these, a positive number divided by a positive number, presents no difficutly. Thus $12\div 3=4$. The example $12\div(-3)=?$ asks us to find a number (the quotient) which multiplied by -3 (the divisor) will give 12 (the dividend). From the work in multiplication it is evident that the unknown number is -4. So $12\div(-3)=-4$. In the same way $(-12)\div(-3)=?$ asks us to find a number which multiplied by -3 will give -12. The number is evidently 4. So $(-12)\div(-3)=4$. Finally $(-12)\div 3=?$ asks us to find a number which multiplied by 3 will give -12 . The number is evidently -4. So $(-12)\div 3=-4$. From the four cases $12\div 3=4$, $12\div(-3)=-4$, $(-12)\div(-3)=4$, and $-12\div 3=-4$ and other similar cases the pupils can form the conclusion:

To divide positive and negative numbers divide their absolute values. If the two numbers have the same sign the quotient is positive. If the two numbers have different signs the quotient is negative.

V. The Fundamental Operations With Algebraic Quantities.

225. Definitions. A quantity such as $3a+7$, and $3x-2y$ involving one or more unknowns represented by letters is called an *algebraic quantity.* The algebraic quantity $3a+2b-7c$ has three terms, namely $3a$, $2b$ and $7c$. That is, the *terms* of an algebraic quantity are the parts of the quantity that are separated from each other by + and − signs. A quantity of one term is called a monomial. $3a$, $4x^2$ and $3xy$ are monomials. A quantity of more than one term is called a *polynomial.* $a+2b$ and $3x+7y-2$ are polynomials.

226. Extent of Work. Some knowledge of operations with algebraic quantities is necessary in order to solve simple equations and to manipulate formulas. For this reason addition, subtraction and multiplication of monomials and polynomials, and division of monomials by monomials and polynomials by monomials should be included in the course.

227. Addition and Subtraction of Monomials. The method of adding and subtracting monomials can be discovered by the pupils from cases such as the following.

$3¢$	3 lbs.	$3a$	$3x$
$5¢$	5 lbs.	$5a$	$5x$
$8¢$	8 lbs.	$8a$	$8x$

It should be noted that only like quantities can be added or subtracted, and the result is like the original quantities, just as in the case of concrete numbers. $3a$ and $5a$ are like quantities and so can be added, the result being $8a$. $3a$ and $2b$ are not like quantities so they cannot be added or combined. All that can be done is to indicate that they are to be added as

$3a+2b$. This indicated addition cannot be performed until the two numbers are both expressed in like terms. If it is found later that $a=5$ and $b=2$, then $3a=15$ and $2b=4$ and $3a+2b=15+4=19$. A similar discussion applies to the subtraction of monomials.

228. Addition and Subtraction of Polynomials. The addition and subtraction of polynomials can be based on the addition and subtraction of two and three place numbers and of compound denominate numbers.

46=4 tens+6 units	$4t+6u$
32=3 tens+2 units	$3t+2u$
78=7 tens+8 units	$7t+8u$
3 ft. 2 in.	$3x+2y$
2 ft. 7 in.	$2x+5y$
5 ft. 9 in.	$5x+7y$

Subtraction may be treated in the same way.

229. Multiplication of Monomials. In the monomials $2x$ $3ab$ and $4y^2$, the 2, 3 and 4 are called the *numerical coefficients* and the x, ab and y^2 are called the *literal factors* of the monomials. The method of multiplying monomials may be derived from cases such as $2a \cdot 3b=2 \cdot a \cdot 3 \cdot b$. Since the result does not depend on the order of the factors ($2 \cdot 3 \cdot 7=2 \cdot 7 \cdot 3=3 \cdot 2 \cdot 7$ etc.) $2 \cdot a \cdot 3 \cdot b=2 \cdot 3 \cdot a \cdot b=6ab$. From this the pupils can see that:

The product of two monomials is the product of their numerical coefficients times the product of their literal factors.

$3b^2 \cdot 4b^3=3 \cdot b \cdot b \cdot 4 \cdot b \cdot b \cdot b=$
$3 \cdot 4 \cdot b \cdot b \cdot b \cdot b \cdot b=12b^5$.

$5xy^2 \cdot 3x^3y^3=5 \cdot x \cdot y \cdot y \cdot 3 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y=$
$5 \cdot 3 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y=15x^4y^5$. That is:

The exponent of each factor in the product equals the sum of the exponents of that factor in the original numbers.

Since the rule for signs in multiplication is already known the pupils can now multiply any two monomials as $-3ab^2 . 4a^4 = -12a^5b^2$

230. Multiplication of a Polynomial by a Monomial. The multiplication of a polynomial by a monomial may be based on cases such as the following:

32	3 tens+2 units	3t+2u	2 ft. 3 in.	2a+ 4b
3	3	3	3	4
96	9 tens+6 units	9t+6u	6 ft. 9 in.	8a+16b

Also: $3 \times (2a+4b) = (2a+4b)+(2a+4b)+(2a+4b) = 6a+12b$. From cases such as these, it is evident that:

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial.

231. Multiplication of a Polynomial by a Polynomial.

31	3 tens+1	3t+1	
22	2 tens+2	2t+2	
62	6 tens+2	6t+2	
62	6 (tens)^2+2 tens	$6t^2+2t$	
682	6 (tens)^2+8 tens+2	$6t^2+8t+2$	

$3a + 2b$
$2a + 4b$
$12ab+8b^2$
$6a^2+ 4ab$
$6a^2+16ab+8b^2$

From these cases it is evident that:

To multiply a polynomial by a polynomial, multiply each term of one by each term of the other and add the products.

232. Division of a Monomial by a Monomial. Division of a monomial by a monomial involves a knowledge of the law of exponents in division. $x^4 \div x^2 = \frac{x \cdot x \cdot x \cdot x}{x \cdot x} = x^2$ since dividing the same factors out of both the numerator and denominator of a fraction does not change its value. In the same way

$a^3b^4 \div a^2b^2 = \frac{a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b}{a \cdot a \cdot b \cdot b} = ab^2$. From such cases as these it is evident that:

The exponent of any factor in the quotient is obtained by subtracting the exponent of that factor in the divisor from its exponent in the dividend.

$$21mn \div 3mn = \frac{3 \cdot 7 \cdot m \cdot n}{3 \cdot m \cdot n} = 7$$

$$6ab \div 2b = \frac{2 \cdot 3 \cdot a \cdot b}{2 \cdot b} = 3a$$

$$15x^4y^3 \div 5x^2y = \frac{3 \cdot 5 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}{5 \cdot x \cdot x \cdot y} = 3x^2y^2$$

From these cases it is seen that:

To divide a monomial by a monomial, divide the numerical coefficient of the dividend by the numerical coefficient of the divisor and each literal factor of the dividend by the same literal factor of the denominator.

Since the law of signs in division is known, the pupils can divide cases such as $(25ab^3c^4) \div (-5ab^2c) = -5bc^3$.

233. Division of a Polynomial by a Monomial.

$3)\overline{96}$	$3)\overline{9 \text{ tens} + 6 \text{ units}}$	$3)\overline{9t + 6u}$	$2)\overline{8 \text{ ft. } 6 \text{ in.}}$	$3)\overline{9x + 6y}$
32	3 tens + 2 units	$3t + 2u$	4 ft. 3 in.	$3x + 2y$

From cases such as these, the pupils draw the conclusion.

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

EXERCISE. FUNDAMENTAL PROCESSES.

1. Add the following.

(1)	(2)	(3)	(4)	(5)	(6)
6	-9	5	$3x^2$	7 ab	$3x + 2y$
-7	2	-8	$8x^2$	$-ab$	$x - 3y$
1	-1	10	$-7x^2$	ab	
5	3	-7	x^2		

(7)	(8)	(9)
$5a+7$	$3x-7y+z$	$x^2-3xy+7y^2$
$2a-1$	$x-y+2z$	$2x^2+5xy-y^2$
$3a-5$	$2x+3y-4z$	

2. Subtract.

(1)	(2)	(3)	(4)	(5)	(6)
-8	$+7$	$-7x$	$-8ab$	$3x^2$	$2x-3y$
$+7$	-9	$+2x$	$-9ab$	$7x^2$	$x+4y$

(7)	(8)
$3x^2+2y^2$	$7a+3b$
$2x^2-y^2$	$9a-5b$

3. Multiply.

(1)	(2)	(3)	(4)	(5)
$+3$	-4	$5a^2$	$-3x^2y$	$-7a^2b^3$
-7	-8	$2a^3$	$-4xy^3$	$4b^5$

(6)	(7)	(8)
$3a-2b$	$5a^2-2a+7$	$3x^2-5xy+y^2$
6	$3a$	$2xy$

4. Divide.

(1)	(2)	(3)	(4)	(5)
9	-6	-24	$8x^3$	$-15x^2y^4$
-3	-2	6	$2x$	$3xy$

(6)	(7)	(8)
$9x^2-6xy$	$8x^3-4x^2+12x$	$6x^2y^2-12y^4$
$3x$	$-4x$	$-6y^2$

CHAPTER XIV.

THE ARITHMETIC OF THE HOME.

I. Introduction.

234. Applications of Arithmetic. In the preceding chapters the subject matter of arithmetic has been reviewed from the point of view of the teacher. In connection with this review many of the simple applications of the subject have been considered. In addition to these, there are a number of applications of arithmetic of a more technical character involving a knowledge of important social situations with which the teacher of arithmetic should be familiar. It is the purpose of this and the following chapters to consider somewhat in detail a few of the more important of these applications.

235. The Finances of the Home. A comparatively small number of the pupils of the elementary schools will engage in business for themselves or become responsible for the finances of a business. Everyone of them, however, girls as well as boys will have to deal with their personal or family finances and it is the duty of the schools to train them for this work. The management of one's personal or family income and expenses is a business, just as much as running a store or operating a factory.

II. Budgets and Accounts.

236. Family Budgets. Most people are compelled to live on a fixed income the amount of which is known a month or a year ahead. Since this is true, the only possibility of saving lies in so planning the expenditures that they will be less than the income. If the money is spent as it is received, with no definite plan, there is usually none left for savings. By care-

fully planning the expenditures for a month or year ahead, however, it is usually possible to lay aside a definite amount as savings each month. Such a predetermined plan of expenditure, based on one's income, is called a *budget.*

In an average family consisting of father, mother and three children, the income should be apportioned about as follows:

Food	25%
Rent	20%
Clothes	15%
Operating expenses	15%
Higher Life	25%

Under Operating Expenses are included light, fuel, wages for help, ice, laundry, etc., and under Higher Life are included insurance, savings, church, books, recreation, health, etc.

The figures given above are averages based on numerous statistics and are given as typical figures which must be varied to meet particular cases. Where the yearly income is less than $1000 a larger per cent than that indicated in the ideal budget given above will ordinarily have to be spent for food. The best guide in making a family budget is the record of past expenditures if such a record is available.

Problems. Family Budgets.

1. A family of two with a yearly income of $2000 have been unable to save and have had nothing to spend for church, books, insurance, etc. They decide to make a budget and try to follow it for the coming year. From their accounts they find that last year they spent for rent $50 a month, for food $387, for clothing $449, for operating expenses $350, and the remainder for recreation. Find what per cent of their income they spent for each of the five items. What items should they try to reduce if possible?

2. A family with a yearly income of $1000 are trying to follow the following budget: 30% for food, 20% for rent, 10% for operating expenses, 15% for clothes and 25% for higher life. At the end of three months they go over their accounts and find that their expenditures have been as follows.

	January	February	March
Rent	$15.00	$15.00	$15.00
Coal	17.50	8.75	8.75
Light	3.15	2.79	2.47
Groceries	20.71	17.32	19.14
Meat	5.24	4.89	5.07
Milk	4.00	4.60	4.80
Telephone	1.25	1.25	1.25
Doctor	.....	4.50	.....
Water	.....	.....	3.00
Clothes	4.50	7.00	12.75
Magazines	.50	.50	.50
Newspapers	.60	.48	.60
Moving Pictures	1.20	1.20	1.20
Church	1.00	1.00	1.00

Classify these expenditures under the five headings of the budget and find what per cent of their income for the three months was spent for each item. What items, if any, exceed the amount provided for in the budget?

3. A high school teacher accepts a position at a salary of $1800 a year. According to the ideal budget given above what is the highest monthly rent he can afford to pay for a house?

237. Personal and Household Accounts. Every individual and family should keep a record of receipts and expenditures. Pupils in the elementary school should be encouraged to keep such accounts as the habit formed here will prove invaluable in after life. Students and teachers who have not already formed this habit should start such accounts at once. The following is a model account for one week.

1921										
Nov.	27	Cash on hand	307	65	Nov.	28	Doctor	2	
	29	John Smith	5			28	Medicine	1	70
Dec.	3	Wages	32	50	Dec.	1	Rent	25	
						3	Groceries	7	15
						3	Meat	3	80
						4	Balance	305	50
			345	15				345	15
Dec.	4	Balance	305	50					

In the form of account given above, the receipts are entered on the left or debit side and the expenditures on the right or credit side. The balance is found by subtracting the sum of the expenditures from the total receipts.

Date 1922	Item	Receipts	Expenditures	Distribution of Expenditures				
				Food	Rent	Clothing	Operating Expenses	Higher Life
Feb. 27	Balance on Hand	$172.18						
Mar. 1	Rent		25.00		25.00			
	Groceries		17.25	17.25				
	Meat		4.81	4.81				
	Electric Lights		3.17				3.17	
	Telephone		1.25				1.25	
2	Moving Pictures		.60					.60
3	Shirt		2.00			2.00		
	Dry Goods		3.72			3.72		
4	Wages	40.00						
	Medicine		1.25					1.25
	Magazine		.35					.35
5	Church		.25					.25
	Totals	$212.18	59.65	22.06	25.00	5.72	4.42	2.45
Mar. 6	Balance on Hand	152.53						

If a family has made a budget, a record of all receipts and expenditures is necessary in order to determine whether or not the budget is being followed. For this purpose the form of account given above is useful. Each amount of money spent

is entered in the column headed Expenditures and also under the proper subdivision of the Distribution of Expenditures. This facilitates the work of determining at any time whether or not the family is living within their budget. If desired the items under Distribution of Expenditures may be subdivided into other items. For example, Operating Expenses might be divided into Heat, Light, etc.

PROBLEMS. ACCOUNTS.

1. The student should buy a small account book and keep a record of all receipts and expenditures, balancing the account each week.

2. Make the following entries in the form of a cash account and balance. Use the first form given.

Feb. 20	Cash on hand	$27.13
	Paid for groceries	2.17
	meat	.40
Feb. 21	Paid for coal 2 tons at $9 a ton	18.00
	groceries	1.73
	fish	.35
	magazine	.25
Feb. 22	Paid for groceries	1.09
	stamps	.25
	shoes	6.00
Feb. 23	Paid for groceries	2.31
	meat	.45
	toilet articles	.75
Feb. 24	Paid for groceries	1.42
	meat	.30
Feb. 25	Paid for meat	.40
	milk tickets	2.00
	paper	.12
	Received salary	45.00

3. Make the following entries in the form of an account and balance. Use the second form given.

Jan. 1	Cash on hand	$117.32
	Paid for rent	30.00
3	Paid for insurance	78.92*
4	Paid for suit	25.00
6	Paid for paper	.12
	milk tickets	5.00
7	Contribution to church	.50
8	Paid for shirt	2.50
9	Paid for concert tickets	4.00
10	Paid for box of candy	1.50
11	Paid barber	.75
12	Paid for book	1.50
	toilet articles	2.10
13	Paid for paper	.12
14	Contribution to church	.50
16	Paid for tickets to basket ball game	.70
18	Paid for shoes	7.00
19	Paid for magazine	.35
20	Paid for phonograph record	1.50
	paper	.12
21	Contribution to church	.50
22	Paid barber	.75
23	Paid for coal	10.50
25	Paid for tickets to moving pictures	.60
26	Paid for candy	.50
27	Paid for paper	.12
28	Contributions to church	.50
29	Paid for medicine	2.00
31	Paid grocery bill	22.32
	meat bill	4.87
	dry goods bill	11.03
	electric light bill	3.12
	gas bill	4.07
	telephone bill	1.25
	Received salary	250.00

*Yearly premium.

4. The items of Problem 3 are the receipts and expenditures of a family for one month. Find what per cent of their monthly income was spent for each of the five items in the budget. If they are trying to live within the budget given below are they succeeding? Food 25%, Rent 20%, Operating Expenses 15%, Clothes 20%, Higher Life 20%.

III. Dealing With a Bank.

238. Kinds of Banks. There are in general two kinds of banks, savings banks and commercial banks. Savings banks will be discussed in the next chapter. Commercial banks ordinarily perform three functions. In the first place, they provide a safe place of deposit for money which may be drawn out as needed. In the second place, they loan money to individuals and business firms at interest. In the third place, they attend

DEPOSITED WITH

THE WOOD COUNTY SAVINGS BANK CO.

By J. C. Smith

Bowling Green, O. Feb. 20 1922.

LIST EACH CHECK SEPARATELY

	Dollars	Cents
Currency	65	
Silver	17	32
Gold		
Checks		
Specify Banks on which drawn		
First Natl. Toledo	25	
State Bank Bowling Green	15	
Total $	122	32

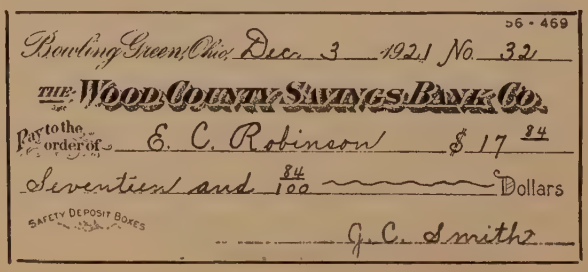
SEE THAT ALL CHECKS AND DRAFTS ARE ENDORSED

to certain business for their customers such as collecting and paying money at a distance, acting as agents in the purchase and sale of various forms of securities, and giving advise on business matters.

239. Opening an Account. The money received for wages or salary should not be kept in the home but should be deposited in a bank for safe keeping. Any responsible person may open an account with a bank by giving references, depositing a sum of money, and leaving a copy of his signature. When the account is opened the depositor is given a bank book in which his deposits are entered and a check book.

240. Making Deposits. To make a deposit, the customer first makes out a *deposit slip* such as that given on preceding page. The deposit slip, bank book, and money are handed in to the teller who enters the amount of the deposit in the bank book.

241. Drawing Money. A customer having money on deposit in a bank may at any time direct the bank to pay any part of his money to himself or any other person. This order to pay out money must be written and signed by the depositor. Such a written order to a bank to pay out money from a deposit is called a *check*. A check should always be written in ink and the amount should be written as far to the left as possible and should be followed by a heavy line so that nothing else can be added. A sample check is given below.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Bowling Green Ohio Dec. 3 1921 No. 32

THE WOOD COUNTY SAVINGS BANK CO.

Pay to the order of E. C. Robinson $17 84/100

Seventeen and 84/100 Dollars

SAFETY DEPOSIT BOXES

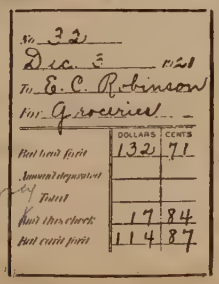
G. C. Smith

When this check is cashed by E. C. Robinson, he must write his name across the back of the left end of the check. This signature on the back known as an *endorsement.* If Mr. Robinson does not cash the check himself, but gives it to a third party, Mr. Jones, he can endorse it as shown below, in which case it can be collected only by Mr. Jones.

Pay to the order of
O. R. Jones
E. C. Robinson

A check may be made out to "Self" in which case it can be collected only by the maker. This is done when the depositor wishes to draw money from the bank for his own use. A check may also be made out to "Order" in which case any one can collect it.

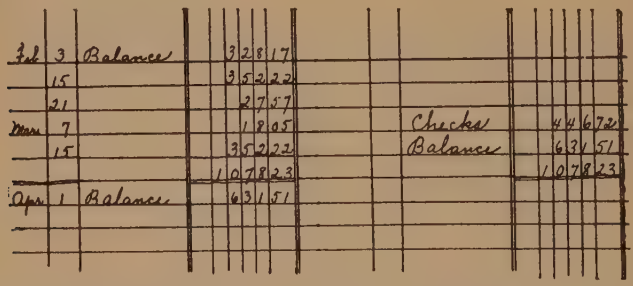
The depositor should keep a record of all checks written, recording the date, number of check, amount, to whom drawn, and for what purpose. For this purpose each blank check is provided with a "stub" from which it may easily be detached. The "stub" is kept by the person writing the check as his record. The stub for the check shown on page 286 is given below. The "stub" also shows the depositor what his balance in the bank is at all times.



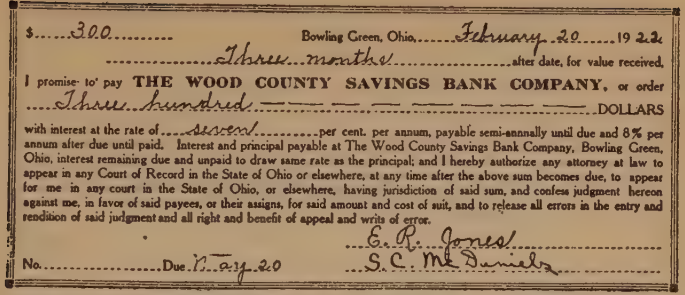
The custom of depositing all money in a bank as soon as received, and of paying all bills by means of checks is growing rapidly. The money is much safer in the bank than in the home and the canceled checks, returned to the maker by the bank, serve as receipts for the bills paid.

Banks do not pay interest on money deposited in this way and subject to withdrawal on demand.

242. Bank Book. When an account is opened the customer receives a bank book in which all deposits are recorded when made by the teller. This book is kept by the depositor and turned into the bank at regular intervals, usually once a month, to be balanced. The book is then returned to the customer together with all checks cashed by the bank since the book was last balanced. The customer should compare the returned checks with his stubs. Below is shown a portion of a bank book giving deposits on the left hand page and withdrawals on the right.



243. Borrowing Money of a Bank. Banks not only provide a safe place for keeping money, but also provide a place where depositors may borrow money in an emergency. Money formerly was borrowed by one individual from another, but today it is usually borrowed from a bank. When money is borrowed of a bank, the borrower gives his written promise to repay the money at a given time and also agrees to pay interest at a given rate for its use. Such a promise to pay is called a *promissory note.* An illustration of such a note is given below.



$ 300 Bowling Green, Ohio, February 20 1922

Three months after date, for value received,

I promise to pay THE WOOD COUNTY SAVINGS BANK COMPANY, or order

Three hundred DOLLARS

with interest at the rate of seven per cent. per annum, payable semi-annually until due and 8% per annum after due until paid. Interest and principal payable at The Wood County Savings Bank Company, Bowling Green, Ohio, interest remaining due and unpaid to draw same rate as the principal; and I hereby authorize any attorney at law to appear in any Court of Record in the State of Ohio or elsewhere, at any time after the above sum becomes due, to appear for me in any court in the State of Ohio, or elsewhere, having jurisdiction of said sum, and confess judgment hereon against me, in favor of said payees, or their assigns, for said amount and cost of suit, and to release all errors in the entry and rendition of said judgment and all right and benefit of appeal and writs of error.

E. R. Jones

No. Due May 20 S. C. McDaniels

When borrowing money of a bank, one must give security for the repayment of the loan. One common form of security is to have one or more responsible persons, in addition to the person borrowing the money, sign the note. This makes them liable for the payment of the note if the borrower is unable to pay. Another form of security is for the borrower to deposit with the bank stocks or bonds of a market value somewhat in excess of the amount of the loan. If the loan is not paid, the bank can sell the stock or bonds and pay themselves out of the proceeds. Thus the owner of a $100 Liberty Loan bond can use it as security in borrowing money of a bank. A third kind of security is a mortgage on real estate owned by the borrower. Such a mortgage gives the bank the right to sell the real estate if the loan is not paid and to pay themselves out of the proceeds.

244. Teaching Pupils About Banks. The pupils of the upper grades of the elementary school should become familiar with the various services that banks render. They should visit the local banks and should have practice in making out deposit slips, checks and notes, and in balancing bank books.

IV. Simple Interest.

245. Meaning of Interest. The pupils in the elementary school all know that if a family makes use of a house that is owned by some one else, they must pay rent for this privilege. Some of the pupils are probably familiar with other cases in

which rent is paid for the use of another's property such as typewriters, pianos, etc. In the same way, if an individual makes use of money belonging to another, he must not only return the money but must also pay for the use of the money. Rent paid for the use of money is called *interest.* The charge for the use of money is usually a certain per cent of the amount, to be paid for its use for one year. The amount of money borrowed is called the *principal* and the per cent paid for its

TABLE OF MAXIMUM AND LEGAL RATES

States and Territories	Legal Rate	Maximum Rate	States and Territories	Legal Rate	Maximum Rate
Ala.	8%	8%	Neb.	7%	10%
Alaska	8%	12%	Nev.	7%	any
Ariz.	6%	12%	N. H.	6%	6%
Ark.	6%	10%	N. J.	6%	6%
Calif.	7%	any	N. Mex.	6%	12%
Colo.	8%	any	N. Y.	6%	6%
Conn.	6%	12%	N. C.	6%	6%
Del.	6%	6%	N. D.	7%	12%
Dist. of Col.	6%	6%	Ohio	6%	8%
Fla.	8%	10%	Okla.	6%	10%
Ga.	7%	8%	Ore.	6%	10%
Hawaii	8%	12%	Pa.	6%	6%
Ida.	7%	12%	P. I.	6%	any
Ill.	5%	7%	P. R.	6%	12%
Ind.	6%	8%	R. I.	6%	any
Ia.	6%	8%	S. C.	7%	8%
Kans.	6%	10%	S. D.	7%	12%
Ky.	6%	6%	Tenn.	6%	6%
La.	5%	8%	Tex.	6%	10%
Me.	6%	any	Utah	8%	12%
Md.	6%	6%	Vt.	6%	6%
Mass.	6%	any	Va.	6%	6%
Mich.	5%	7%	Wash.	6%	12%
Minn.	6%	10%	W. Va.	6%	6%
Miss.	6%	8%	Wis.	6%	10%
Mo.	6%	8%	Wyo.	8%	12%
Mont.	8%	any			

use for one year is called the *rate*. Thus if $250 is borrowed at 6% interest, it means that 6% of $250 must be paid for the use of the $250 for one year. The $250 is the principal and the 6% is the rate. It is evident that the interest for one year is always the product of the rate and the principal.

246. Maximum and Legal Rates. Most states have found it necessary to limit interest rates by fixing a *maximum rate* which can not be exceeded. Most states also fix a rate, known as the *legal rate* to apply in all cases in which it is known by law or by agreement between the parties that interest is to be paid but in which there is no agreement as to the rate. Thus a note reading "with interest" but not specifying the rate bears interest at the legal rate.

247. Factors Determining Interest Rates. Although the law sets certain limits on interest rates the exact rate charged varies within these limits. The chief factor determining interest rates is the relationship between the supply of money available and the demand for money. Other things being equal, if the supply exceeds the demand, interest rates are low, whereas, if the demand is greater than the supply, the rates are high. The security given is another factor affecting the rate of interest. In general interest rates are higher when the security is poor than when it is good. Thus the United States Government can borrow money at a lower rate than an individual or another government more insecure or having fewer resources.

The amount of the loan and the time for which the money is borrowed also sometimes affect the rate. In general, the rate is apt to be higher on small loans or for short periods than on larger loans or for longer periods as the expenses incidental to loaning money in small amounts and for short periods are considerably greater and some interest is lost as a portion of the money is usually on hand waiting to be loaned.

248. The General Method of Computing Interest. It has already been seen from the definition of interest that the interest

on any principal for one year is the product of the rate and the principal. Thus the interest on \$785 at 6% for one year is 6% of \$785=.06×\$785=\$47.10. For three years the interest is evidently 3×.06×\$785=\$141.30. It is evident that, in general:

Interest is the product of the time, the rate and the principal, or $I=T\times R\times P$.

It should be noticed that the time must be expressed in years and that the rate is a fraction as .06 or $\frac{6}{100}$ and not simply 6.

At the present time money is usually borrowed from banks and for less than a year and even when the loan is for several years the interest must be paid yearly. When the loan is for less than a year, it is usually for periods of 30 days, 60 days, etc., or for 1 month, 2 months, etc. A note payable in 30 days is due on the thirtieth day following the date of the note. Thus if the note is dated December 17, it will fall due on January 16. On the other hand a note payable in 1 month is due the same date of the following month. Thus if dated December 17, such a note will fall due on January 17.

EXERCISE. SIMPLE INTEREST.

Find the interest on each of the following using the *General Method.*

1. \$2500 at 6% for 30 days.

Solution.

$$I=\frac{\cancel{30}}{\cancel{360}_{12}}\times\frac{\cancel{6}}{\cancel{100}}\times\overset{25}{\cancel{2500}}=2\tfrac{5}{2}=\$12.50$$

2. \$875 at 7% for 6 months.
3. \$5870 at 5% for 9 months.
4. \$1000 at 8% for 1 month.
5. \$350 at 7% for 90 days.
6. \$3890 at 4% for 3 years.

249. Methods of Finding the Time. It is occasionally necessary to compute interest when the time is not specified as 60 days or 3 months but must be determined by finding the time that has elapsed from the date of the note to the date on which the interest is paid. Most states count one of the two end dates in figuring the time elapsed, although some count neither and some both of these dates. Two methods of finding the time are in rather common use.

Example. Find the time from May 23, to November 9 of the same year.

First Method: By compound subtraction of dates.

11	9
5	23
5 mo.	16 da. = 166 days = $^{166}/_{360}$ of a year.

In figuring interest it is customary to regard the year as composed of 12 months of 30 days each, or of 360 days.

Second Method: By counting the exact number of days.

8	days	in	May (after May 23)
30	"	"	June
31	"	"	July
31	"	"	August
30	"	"	September
31	"	"	October
9	"	"	November
170	days = $^{17}/_{360}$ of a year.		

The labor of finding the exact number of days can be greatly reduced by the use of a table such as the one given on the next page. The method of using the table should be clear from the following examples.

Example 1. Find the number of days from September 7 to June 12 of the following year.

Solution.

From the table we find that from September 7 to June 7 is 273 days. From June 7 to June 12 is 5 days, so the entire time is 273+5=278 days.

Example 2. Find the number of days from March 18 to October 7 of the same year.

Solution. From March 18 to October 18 is, from the table, 214 days. October 18 is 11 days after October 7, so the time desired is 214—11=203 days.

Table Showing the Number of Days from Any Day of Any Month to the Same Day of Any Other Month Not More than One Year Later.

From Any Day of	To the Same Day of the Next											
	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
January	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
August	153	184	212	243	273	304	334	365	31	61	92	122
September ...	122	153	181	212	242	273	303	334	365	30	61	91
October	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	212	243	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

EXERCISE. SIMPLE INTEREST.

I. Find the time between the dates given below by compound subtraction. Change to fractional part of year, calling the year 360 days.

1. From Feb. 21 to July 27.
2. From Dec. 7 to March 10.
3. From April 18 to July 1.
4. From August 29 to Dec. 3.
5. From July 7 to Sept. 1.

II. Find the number of days between the dates given in I above by use of the table on page 294.

III. Find the interest and amount due on each of the notes given below. Find the exact number of days in each case by use of the table and find the interest by the General Method.

Principal	Rate of Interest	Date of Note	Date Paid
1. $5000	7%	Dec. 7, 1921	Feb. 1, 1922
2. $3500	7%	Jan. 13	April 7
3. $780	6%	Aug. 21	Nov. 8
4. $7500	6½%	March 8	July 30
5. $900	5%	April 15	June 7

250. Interest Tables. In banks and other places where interest is constantly computed, interest tables are often used to lessen the labor. Interest tables are of various kinds to suit different conditions. One form of table is given on page 297. The method of using the table should be clear from the following examples.

Example 1. Find the interest on $3760 for 90 days at 6%.

Solution.

Int. on $3000 for 90 days (3 months)=$45.00

(This is found by multiplying the interest on $300 by 10)

Int. on \$700 for 90 days	= 10.50
Int. on \$60 for 90 days	= .90
(This is found by dividing the interest on \$600 by 10)	
	\$56.40

A complete set of tables usually consists of tables similar to the one given here, each giving the interest at different rates such at 3%, 3½%, 4%, 4½%, etc. It is possible, however, to find the interest at different rates by using the 6% table given here.

Example 2. Find the interest on \$5800 for 47 days at 7%.
Solution.

47 days=1 month, 17 days.

Int. on \$5000 for 1 month=	\$25.00
Int. on \$ 800 for 1 month=	4.00
Int. on \$5000 for 17 days=	14.17
Int. on \$ 800 for 17 days=	2.27
Interest at 6%=	45.44
Interest at 1%=	7.57
(⅙ of interest at 6%)	
Interest at 7%=	\$53.01

EXERCISE. FINDING INTEREST BY TABLES.

I. Tell how to get the interest at each of the following rates by first finding the interest at 6% using the table on page 297, and then changing to the required rate.

1. 4% 2. 4½% 3. 5% 4. 7% 5. 7½% 6. 8%

Hint. To find interest at 4½%, divide the interest at 6% by 4, this will be the interest at 1½%. Subtract this from the interest at 6% and it will leave the interest at 4½%.

INTEREST TABLE. INTEREST AT 6%

Da.	$100	$200	$300	$400	$500	$600	$700	$800	$900	$1000	Da.
1	0.017	0.033	0.050	0.067	0.083	0.100	0.117	0.133	0.150	0.167	1
2	0.033	0.067	0.100	0.133	0.167	0.200	0.233	0.267	0.300	0.333	2
3	0.050	0.100	0.150	0.200	0.250	0.300	0.350	0.400	0.450	0.500	3
4	0.067	0.133	0.200	0.267	0.333	0.400	0.467	0.533	0.600	0.667	4
5	0.083	0.167	0.250	0.333	0.417	0.500	0.583	0.667	0.750	0.833	5
6	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000	6
7	0.117	0.233	0.350	0.467	0.583	0.700	0.817	0.933	1.050	1.167	7
8	0.133	0.267	0.400	0.533	0.667	0.800	0.933	1.067	1.200	1.333	8
9	0.150	0.300	0.450	0.600	0.750	0.900	1.050	1.200	1.350	1.500	9
10	0.167	0.333	0.500	0.667	0.833	1.000	1.167	1.333	1.500	1.667	10
11	0.183	0.367	0.550	0.733	0.917	1.100	1.283	1.467	1.650	1.833	11
12	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000	12
13	0.217	0.433	0.650	0.867	1.083	1.300	1.517	1.733	1.950	2.167	13
14	0.233	0.467	0.700	0.933	1.167	1.400	1.633	1.867	2.100	2.333	14
15	0.250	0.500	0.750	1.000	1.250	1.500	1.750	2.000	2.250	2.500	15
16	0.267	0.533	0.800	1.067	1.333	1.600	1.867	2.133	2.400	2.667	16
17	0.283	0.567	0.850	1.133	1.417	1.700	1.983	2.267	2.550	2.833	17
18	0.300	0.600	0.900	1.200	1.500	1.800	2.100	2.400	2.700	3.000	18
19	0.317	0.633	0.950	1.267	1.583	1.900	2.217	2.533	2.850	3.167	19
20	0.333	0.667	1.000	1.333	1.667	2.000	2.333	2.667	3.000	3.333	20
21	0.350	0.700	1.050	1.400	1.750	2.100	2.450	2.800	3.150	3.500	21
22	0.367	0.733	1.100	1.467	1.833	2.200	2.567	2.933	3.300	3.667	22
23	0.383	0.767	1.150	1.533	1.917	2.300	2.683	3.067	3.450	3.833	23
24	0.400	0.800	1.200	1.600	2.000	2.400	2.800	3.200	3.600	4.000	24
25	0.417	0.833	1.250	1.667	2.083	2.500	2.917	3.333	3.750	4.167	25
26	0.433	0.867	1.300	1.733	2.167	2.600	3.033	3.467	3.900	4.333	26
27	0.450	0.900	1.350	1.800	2.250	2.700	3.150	3.600	4.050	4.500	27
28	0.467	0.933	1.400	1.867	2.333	2.800	3.267	3.733	4.200	4.667	28
29	0.483	0.967	1.450	1.933	2.417	2.900	3.384	3.867	4.350	4.833	29
Mo.	**$100**	**$200**	**$300**	**$400**	**$500**	**$600**	**$700**	**$800**	**$900**	**$1000**	**Mo.**
1	0.500	1.000	1.500	2.000	2.500	3.000	3.500	4.000	4.500	5.000	1
2	1.000	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000	2
3	1.500	3.000	4.500	6.000	7.500	9.000	10.500	12.000	13.500	15.000	3
4	2.000	4.000	6.000	8.000	10.000	12.000	14.000	16.000	18.000	20.000	4
5	2.500	5.000	7.500	10.000	12.500	15.000	17.500	20.000	22.500	25.000	5
6	3.000	6.000	9.000	12.000	15.000	18.000	21.000	24.000	27.000	30.000	6
7	3.500	7.000	10.500	14.000	17.500	21.000	24.500	28.000	31.500	35.000	7
8	4.000	8.000	12.000	16.000	20.000	24.000	28.000	32.000	36.000	40.000	8
9	4.500	9.000	13.500	18.000	22.500	27.000	31.500	36.000	40.500	45.000	9
10	5.000	10.000	15.000	20.000	25.000	30.000	35.000	40.000	45.000	50.000	10
11	5.500	11.000	16.500	22.000	27.500	33.000	38.500	44.000	49.500	55.000	11
Yr.	**$100**	**$200**	**$300**	**$400**	**$500**	**$600**	**$700**	**$800**	**$900**	**$1000**	**Yr.**
1	6.00	12.00	18.00	24.00	30.00	36.00	42.00	48.00	54.00	60.00	1
2	12.00	24.00	36.00	48.00	60.00	72.00	84.00	96.00	108.00	120.00	2
3	18.00	36.00	54.00	72.00	90.00	108.00	126.00	144.00	162.00	180.00	3
4	24.00	48.00	72.00	96.00	120.00	144.00	168.00	192.00	216.00	240.00	4
5	30.00	60.00	90.00	120.00	150.00	180.00	210.00	240.00	270.00	300.00	5

II. Find the interest and amount due on each of the notes given below. Find the exact number of days by means of table on page 294, and find the interest by means of the table on page 297.

Principal	Rate of Interest	Date of Note	Date Paid
1. $4000	6%	Jan. 13	Feb. 1
2. $7000	6%	Feb. 27	Aug. 12
3. $2500	6%	June 7	Nov. 20
4. $10,000	7%	Sept. 15	Dec. 9
5. $5000	8%	April 15	July 29
6. $3700	5%	March 3	May 17
7. $8590	7½%	July 11	Oct. 1
8. $32,500	4½%	Aug. 21	Sept. 8

251. Bank Discount. Many banks require that the interest on loans be paid in advance. In such cases the note is made out without interest and the borrower receives the face value of the note less the interest. Interest deducted in advance in this manner is called *bank discount.* The amount received by the borrower, that is the face of the note less the interest, is called the *proceeds.*

EXERCISE. BANK DISCOUNT.

I. On February 26, Mr. A. R. Jones borrowed $3500 at his bank for 90 days. How much did he receive, the rate of discount being 7%? When will the note fall due? How much will he have to pay the bank at that time?

II. Find the discount and proceeds in each of the following cases. Use the table in finding the discount.

Principal	Rate of Discount	Time for which Money is Borrowed
1. $300	6%	60 days
2. $5000	7%	90 days
3. $7500	7%	4 months
4. $900	8%	20 days
5. $1000	7½%	6 months

V. Buying Goods at Discount.

252. Economy of Purchase. In order to keep within a budget and save money on a fixed income, it is necessary to exercise economy in making purchases. Besides using good judgment in purchasing goods and paying enough to get a quality that will give service, there are three principal ways of economizing on purchases, by paying cash, by buying staples in quantities, and by taking advantage of sales.

253. Paying Cash. A store that does a credit business must necessarily charge higher prices than a strictly cash store. In the first place, there is always a small percentage of loss due to failure or inability of customers to pay accounts. This loss is inevitable, by using great care in giving credit a store can cut down this loss to a small percentage of the total amount of business done, but can not wipe it out altogether. From past experience, a credit store knows just about what this loss will amount to in a year and takes it into consideration in fixing the prices, that is the prices are made high enough to cover this loss.

In the second place, it takes more capital to run a credit business. If instead of receiving the money at the time the sale is made the store must wait say 90 days, it loses the use of this money for that time. To make up for this, it must add enough to its prices to pay the interest on the amount due for the time it must wait.

In the third place, a cash store can often buy more cheaply as it is in a better position to pay cash itself when buying its goods and so can buy more cheaply.

It is clear then that the purchaser can save money by paying cash and buying of a store that sells for cash only or that makes a different price for cash and credit. Some stores that do both a cash and credit business make no distinction and charge the same whether cash is paid or credit is extended.

Where this is done the prices must be a compromise between cash and credit prices and the cash customer is helping the credit customer pay for the privilege of receiving credit. On the other hand, many stores that do both a cash and a credit business make a lower price for cash. This is commonly done in one of two ways. Some stores of this kind mark all goods at the credit price and give a reduction or a *discount* for cash. Other stores mark goods at the cash price and if credit is extended, charge interest on the account until paid.

Stores such as groceries, where the individual purchases are small, sometimes give a discount for cash by means of cash register slips. When cash is paid, the customer is given a printed slip showing the amount of his purchase. When he has saved slips totaling a certain amount, say ten dollars, the customer can return them and receive say fifty cents for them in trade. This amounts to allowing a 5% discount for cash.

254. Buying in Quantities. Money can be saved by buying staple goods such as canned goods, sugar, potatoes, flour, soap, etc., in quantities rather than in small amounts as needed. The cost of weighing, wrapping, and delivering a small purchase is almost as much as for a large one. The cost of selling and delivering a bushel of potatoes all at one time is much less than the cost of selling and delivering the same amount a few pounds at a time. The merchant must, therefore, put a higher comparative price on goods sold in small quantities.

255. Sales. The customer can also often save money by taking advantage of the sales held by stores at various times during which a reduction in prices is made. There are a variety of reasons for holding these sales. Many kinds of goods are seasonal, that is they are sold only during certain seasons of the year. Many items of clothing such as suits, coats, overcoats, underwear, and furs; and such goods as stoves, refrigerators, porch swings, etc., belong in this class. For a variety of reasons it is usually poor business to carry seasonal goods

in stock from one season to the next. Most merchants, therefore, hold end of season sales in which they sell all such goods at a price just sufficient to cover the cost of the goods and the expense of handling them, sacrificing all or part of their profit. If the goods were kept in stock until the next season they might in some cases go out of style or become damaged and shopworn. It is particularly dangerous to carry the goods over if there is any chance of prices falling as the merchant might have to sell them the next year at an actual loss. Further the merchant usually needs the space occupied by the out of season goods for his new stock and he needs the money tied up in such goods to pay for the new stock. Finally, the merchant makes his profit by keeping his capital constantly at work, by buying goods and selling at a profit and then buying and selling again. It is poor business, therefore, to have money tied up in goods for which there will be no sale for a year, it is much better to get the money out even without a profit and to put it in other goods which can be sold at a profit within a reasonable time.

Many stores also make a practice of holding clearance sales at regular intervals. There are several reasons for such sales. Such sales are often held in order to reduce the stock before taking an inventory. Any business must at regular intervals make a list of the stock in hand and determine its value. Such a summary of stock on hand is called an *inventory*. The work and expense of making an inventory is great so it is customary to reduce the stock as much as possible before starting. Clearance sales also serve to get rid of shopworn goods, and all goods that is not selling for any reason. As has already been considered, it is poor business to keep money tied up in goods, if it will not sell at a profit it must be gotten rid of without profit so that the money may be put in more profitable goods. Clearance sales also serve to get rid of incomplete lines. An up-to-date shoe store, for example, tries to keep a complete range of sizes in every style of shoe sold. Whenever certain sizes of a given style are sold out the store either buys more, or

if it wants to stop selling that style tries to sell out the remaining sizes as quickly as possible.

Some stores hold weekly or monthly sales for advertising purposes. The Friday bargain sales held by some department stores are of this kind. The purpose of these sales is to make new customers for the store. By sacrificing part or all profit on certain articles on these bargain days customers are drawn to the store and in this way get acquainted with the store and form the habit of trading there. Certain articles are sometimes sold at an actual loss at such sales and the loss charged to the fund the store sets aside each year for advertising. A certain drug firm having a chain of stores all over the country holds at regular intervals what are known as one cent sales. During these sales any customer buying an article at the regular price has the privilege of buying a second article of the same kind for one cent. These sales are held in order to advertise these stores and their products.

While many sales are held for legitimate reasons, there are other sales which are dishonest in their intent. Pupils in the elementary schools should study the advertising of the local stores and become familiar with some of the legitimate reasons for holding sales.

Problems. Economy of Purchase.

1. A furniture store marks all their goods to sell on the following terms: $\frac{1}{3}$ down and the balance in ten equal installments. If cash is paid, they allow a discount of 20% from the marked price. Find the cash price on each of the following, and the amount saved by paying cash.

a. 8 piece dining room suit, marked $260.
b. 2 piece upholstered davenport suite, marked $185.
c. Rocker, marked $22.
d. Library table, marked $42.50.

2. A certain make of sewing machine is listed to sell on payments of $5 a month. If cash is paid a discount of 20% of the list price is allowed. Find the cash price on each of the following models and the amount saved by paying cash.

a. Model A, listed at $90.

b. Model B, listed at $86.

3. A certain make of talking machine is listed to sell for cash. If the customer prefers, however, he may pay $10 down and $10 a month with 6% interest on all deferred payments. The company deducts the interest on deferred payments from the $10 paid each month and credits the rest to the principal. Find how much you would pay altogether for an instrument listed at $150 if you bought it on the deferred payment plan. How much could you save by paying cash?

4. A coal company advertises a discount of 12½¢ a ton on all orders for two tons or over. What would you save on two tons of Pocahontas lump at $7.90 per ton by buying it all at one time? What per cent is the saving of the total price figured at $7.90 a ton?

5. If potatoes are selling at 3¢ a pound in small lots, at $1.75 a bushel, or at $4.00 for a bag containing 2½ bushels, what per cent of the total price figured at 3¢ a pound do you save by buying a bushel? By buying a bag? (1 bu.=60 lbs.)

6. Find the per cent of the total price saved on each of the following by buying a box of 12 cans.

a. Peaches 50¢ a can, or $5.50 a box.

b. Corn 25¢ a can, or $2.70 a box.

c. Peas 25¢ a can, or $2.75 a box.

7. A grocery store sells a certain kind of soap at 10¢ a cake, or 3 cakes for 25¢. Find the per cent of the total price, figured at 10¢ a cake, that you save by buying 3 cakes.

8. A dry goods store sells a certain grade of unbleached muslin at 13¢ a yard. If you buy a whole bolt they charge only 12¢ a yard. What per cent can you save by buying a bolt?

9. A certain brand of starch sells at 10¢ a pound package, or 3 for 28¢, or 5 for 45¢. What per cent of the total price, figured at 10¢ a pound, can you save by buying 3 boxes? By buying 5 boxes?

10. Do you think of any reasons for holding sales besides these mentioned in the text?

11. The statement is made in the text that some sales are dishonest in their intent. Can you give illustrations of this?

12. A clothing store during its January clearance sale offers a 20% discount on any suit or overcoat in stock. What might be some of the reasons for offering the discount? What would you have to pay for a suit marked $40?

13. A store advertises *Positive Savings of* 25% *to* 50% on all women's wearing apparel. Included in the advertisement is the following:

$39.50	Plush	coats	reduced	to.....	$29.75
49.50	"	"	"	"	34.75
65.00	"	"	"	"	49.75
75.00	"	"	"	"	59.75
87.50	"	"	"	"	69.75
100.00	"	"	"	"	74.75

Find the per cent of discount on each coat. Is the advertisement truthful?

14. In January a store advertises a discount of 33⅓% on entire stock of furs. What might be some of the reasons for this? What would you have to pay for a 36-inch coney coat marked $65?

15. A clothing store advertises a Pre-inventory sale offering discounts ranging from 20% to 50%. What are the reasons for holding such a sale? They advertise $65 suits at $43.34. What per cent discount is this?

CHAPTER XV.

THE ARITHMETIC OF SAVINGS AND INVESTMENT.

I. The Accumulation and Safe Keeping of Savings.

256. Savings Institutions. In the preceding chapter the making and following of a budget and the keeping of a record of receipts and expenditures were discussed as necessary steps towards the saving of money. As soon as savings start the question arises as to what to do with them. We have in this country several institutions that not only provide a safe place for the deposit of savings but that also encourage the habit of saving. The most important of these are the private Savings Banks and the Government Postal Savings Banks.

257. Importance of Savings Instruction in the Schools. The various savings institutions are doing much towards the education of adults in saving. The most effective place to teach saving, however, is in the elementary schools. "The logical starting-place of savings-banks is the elementary schools. Savings-banks should above all be placed within the knowledge of children, and their methods and advantages should be explained to every child. Savings-banks offer most attractive facilities for impressing youth with the value of providence. While the pupil is learning the advantages of sacrificing the small pleasures of the present for the greater pleasures of the future through his weekly investments in savings-bank credits, he is acquiring a zest for the mastery of arithmetical sums and the rules of interest."*

A number of schools have tried the experiment of a school savings bank, in co-operation with a local savings institution or with the Postal Savings Bank. The curriculum of every

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*From Hamilton's Savings and Savings Institutions. Used by permission of The Macmillan Company, publishers.

school should provide regular instruction in savings and should encourage the pupils to start savings accounts. The savings bank is the best education in practical economy. The teacher should familiarize herself and her pupils with the facilities and methods for savings offered by local institutions and by the government Postal Savings Bank, she has no more important duty than this and can perform no greater service to society.

258. Compound Interest. Most savings institutions pay *compound interest* on deposits. That is, the interest when due is added to the principal and becomes part of the principal for the next interest period. The interest may be added to the principal once a year, twice a year, or four times a year. In these cases the interest is said to be compounded annually, semi-annually and quarterly, respectively. The following example illustrates the principle of compound interest.

Problem. Find the amount at the end of the year of $500 deposited in a savings bank on January 1 at 4% per annum compounded quarterly.

Solution. 4% per annum is 1% per quarter. The interest each quarter is, therefore, just $\frac{1}{100}$ of the principal and may be found by shifting the decimal point.

$500.00	
5.00	Interest first quarter.
505.00	
5.05	Interest second quarter.
510.05	
5.10	Interest third quarter.
515.15	
5.15	Interest fourth quarter.
520.30	Amount at end of year.

The statement is frequently made that compound interest is illegal. Such a statement is misleading. There are no laws against the *paying* of compound interest. In the majority of states, however, the collection of compound interest can not be enforced by law.

259. Compound Interest Tables. The preceding example serves to illustrate the meaning of compound interest and the student should solve several problems in this way to make sure that the meaning is clear. In practice, compound interest is usually computed by the aid of tables such as given on pages 308 and 309.

Problem. Find the amount of \$300 at the end of 4 years, if deposited at 5% per annum compounded semi-annually.

Solution. 5% per annum equals 2½% semi-annually. As the interest is compounded semi-annually there are 8 interest periods. The problem is, therefore to find the amount of \$1 at the end of 8 interest periods at 2½% for the period. From the table this is found to be 1.21840290. The amount of \$300 will be 300 times this or \$365.52.

EXERCISES AND PROBLEMS. COMPOUND INTEREST.

1. Without using the table find the amount of each of the following.

a. \$1500 at the end of 3 years at 4½% compounded annually.

b. \$2150 at the end of 2 years at 5% compounded semi-annually.

c. \$750 at the end of 2 years at 4% compounded quarterly.

2. Using the table find the amount of each of the following.

a. \$3000 at the end of 10 years at 5% compounded annually.

b. \$3000 at the end of 10 years at 5% compounded semi-annually.

c. \$3000 at the end of 10 years at 6% compounded quarterly.

COMPOUND INTEREST TABLE.

Showing the amount of $1.00 at compound interest at various rates, for any number of interest periods up to 50.

Period	Rate for Period						
	1%	1½%	2%	2½%	3%	3½%	4%
1	1.0100 000	1.0150 000	1.0200 0000	1.0250 0000	1.0300 0000	1.0350 0000	1.0400 0000
2	1.0201 000	1.0302 250	1.0404 0000	1.0506 2500	1.0609 0000	1.0712 2500	1.0816 0000
3	1.0303 010	1.0456 784	1.0612 0800	1.0768 9062	1.0927 2700	1.1087 1787	1.1248 6400
4	1.0406 040	1.0613 636	1.0824 3216	1.1038 1289	1.1255 0881	1.1475 2300	1.1698 5856
5	1.0510 101	1.0772 840	1.1040 8080	1.1314 0821	1.1592 7407	1.1876 8631	1.2166 5290
6	1.0615 202	1.0934 433	1.1261 6242	1.1596 9342	1.1940 5230	1.2292 5533	1.2653 1902
7	1.0721 354	1.1098 450	1.1486 8567	1.1886 8575	1.2298 7387	1.2722 7926	1.3159 3178
8	1.0828 567	1.1264 926	1.1716 5938	1.2184 0290	1.2667 7008	1.3168 0904	1.3685 6905
9	1.0936 853	1.1433 900	1.1950 9257	1.2488 6297	1.3047 7318	1.3628 9735	1.4233 1181
10	1.1046 221	1.1605 408	1.2189 9442	1.2800 8454	1.3439 1638	1.4105 9876	1.4802 4428
11	1.1156 683	1.1779 489	1.2433 7431	1.3120 8666	1.3842 3387	1.4599 6972	1.5394 5406
12	1.1268 250	1.1956 182	1.2682 4179	1.3448 8882	1.4257 6089	1.5110 6866	1.6010 3222
13	1.1380 933	1.2135 624	1.2936 0663	1.3785 1104	1.4685 3371	1.5639 5606	1.6650 7351
14	1.1494 742	1.2317 557	1.3194 7876	1.4129 7382	1.5125 8972	1.6186 9452	1.7316 7645
15	1.1609 690	1.2502 321	1.3458 6834	1.4482 9817	1.5579 6742	1.6753 4883	1.8009 4351
16	1.1725 786	1.2689 855	1.3727 8570	1.4845 0562	1.6047 0644	1.7339 8601	1.8729 8125
17	1.1843 044	1.2880 203	1.4002 4142	1.5216 1826	1.6528 4763	1.7946 7555	1.9479 0050
18	1.1961 475	1.3073 406	1.4282 4625	1.5596 5872	1.7024 3306	1.8574 8920	2.0258 1652
19	1.2081 090	1.3269 507	1.4568 1117	1.5986 5019	1.7535 0605	1.9225 0132	2.1068 4918
20	1.2201 900	1.3468 550	1.4859 4740	1.6386 1644	1.8061 1123	1.9897 8886	2.1911 2314
21	1.2323 919	1.3670 578	1.5156 6634	1.6795 8185	1.8602 9457	2.0594 3147	2.2787 6807
22	1.2447 159	1.3875 637	1.5459 7967	1.7215 7140	1.9161 0341	2.1315 1158	2.3699 1879
23	1.2571 630	1.4083 772	1.5768 9926	1.7646 1068	1.9735 8651	2.2061 1448	2.4647 1555
24	1.2697 346	1.4295 028	1.6084 3725	1.8087 2595	2.0327 9411	2.2833 2849	2.5633 0417
25	1.2824 320	1.4509 454	1.6406 0599	1.8539 4410	2.0937 7793	2.3632 4498	2.6658 3633
26	1.2952 563	1.4727 095	1.6734 1811	1.9002 9270	2.1565 9127	2.4459 5856	2.7724 6979
27	1.3082 089	1.4948 002	1.7068 8648	1.9478 0002	2.2212 8901	2.5315 6711	2.8833 6858
28	1.3212 910	1.5172 222	1.7410 2421	1.9964 9502	2.2879 2768	2.6201 7196	2.9987 0332
29	1.3345 039	1.5399 805	1.7758 4469	2.0464 0739	2.3565 6551	2.7118 7798	3.1186 5145
30	1.3478 490	1.5630 802	1.8113 6158	2.0975 6758	2.4272 6247	2.8067 9370	3.2433 9751
31	1.3613 274	1.5865 264	1.8475 8882	2.1500 0677	2.5000 8035	2.9050 3148	3.3731 3341
32	1.3749 407	1.6103 243	1.8845 4059	2.2037 5694	2.5750 8276	3.0067 0759	3.5080 5875
33	1.3886 901	1.6344 792	1.9222 3140	2.2588 5086	2.6523 3524	3.1119 4235	3.6483 8110
34	1.4025 770	1.6589 964	1.9606 7603	2.3153 2213	2.7319 0530	3.2208 6033	3.7943 1634
35	1.4166 028	1.6338 813	1.9998 8955	2.3732 0519	2.8138 6245	3.3335 9045	3.9460 8899
36	1.4307 688	1.7091 395	2.0398 8734	2.4325 3532	2.8982 7833	3.4502 6611	4.1039 3255
37	1.4450 765	1.7347 766	2.0806 8509	2.4933 4870	2.9852 2668	3.5710 2543	4.2680 8986
38	1.4595 272	1.7607 983	2.1222 9879	2.5556 8242	3.0747 8348	3.6960 1132	4.4388 1345
39	1.4741 225	1.7872 103	2.1647 4477	2.6195 7448	3.1670 2698	3.8253 7171	4.6163 6599
40	1.4888 637	1.8140 184	2.2080 3967	2.6850 6384	3.2620 3779	3.9592 5972	4.8010 2063
41	1.5037 524	1.8412 278	2.2522 0046	2.7521 9043	3.3598 9893	4.0978 3381	4.9930 6145
42	1.5187 899	1.8688 471	2.2972 4447	2.8209 9520	3.4606 9589	4.2412 5799	5.1927 8391
43	1.5339 778	1.8968 798	2.3431 8936	2.8915 2008	3.5645 1677	4.3897 0202	5.4004 9527
44	1.5493 176	1.9253 330	2.3900 5315	2.9638 0808	3.6714 5227	4.5433 4160	5.6165 1508
45	1.5648 107	1.9542 130	2.4378 5421	3.0379 0328	3.7815 9584	4.7023 5855	5.8411 7568
46	1.5804 589	1.9835 262	2.4866 1129	3.1138 5086	3.8950 4372	4.8669 4110	6.0748 2271
47	1.5962 634	2.0132 791	2.5363 4351	3.1916 9713	4.0118 9503	5.0372 8404	6.3178 1562
48	1.6122 261	2.0434 783	2.5870 7039	3.2714 8956	4.1322 5188	5.2135 8898	6.5705 2824
49	1.6283 483	2.0741 305	2.6388 1179	3.3532 7680	4.2562 1944	5.3960 6459	6.8333 4937
50	1.6446 318	2.1052 424	2.6915 8803	3.4371 0872	4.3839 0602	5.5849 2686	7.1066 8335

COMPOUND INTEREST TABLE.

Showing the amount of $1.00 at compound interest at various rates, for any number of interest periods up to 50.

Period	Rate for Period						
	4½ %	5%	6%	7%	8%	9%	10%
1	1.0450 0000	1.0500 000	1.0600 000	1.0700 000	1.0800 000	1.0900 000	1.1000 000
2	1.0920 2500	1.1025 000	1.1236 000	1.1449 000	1.1664 000	1.1881 000	1.2100 000
3	1.1411 6612	1.1576 250	1.1910 160	1.2250 430	1.2597 120	1.2950 290	1.3310 000
4	1.1925 1860	1.2155 063	1.2624 770	1.3107 960	1.3604 890	1.4115 816	1.4641 000
5	1.2461 8194	1.2762 816	1.3382 256	1.4025 517	1.4693 281	1.5386 240	1.6105 100
6	1.3022 6012	1.3400 956	1.4185 191	1.5007 304	1.5668 743	1.6771 001	1.7715 610
7	1.3608 6183	1.4071 004	1.5036 303	1.6057 815	1.7138 243	1.8280 391	1.9487 171
8	1.4221 0061	1.4774 554	1.5938 481	1.7181 862	1.8509 302	1.9925 626	2.1435 888
9	1.4860 9514	1.5513 282	1.6894 790	1.8384 592	1.9990 046	2.1718 933	2.3579 477
10	1.5529 6942	1.6288 946	1.7908 477	1.9671 514	2.1589 250	2.3673 637	2.5937 425
11	1.6228 5305	1.7103 394	1.8982 986	2.1048 520	2.3316 390	2.5804 264	2.8531 167
12	1.6958 8143	1.7958 563	2.0121 965	2.2521 916	2.5181 791	2.8126 648	3.1384 284
13	1.7721 9610	1.8856 491	2.1329 283	2.4098 450	2.7196 237	3.0658 046	3.4522 712
14	1.8519 4492	1.9799 316	2.2609 040	2.5785 342	2.9371 936	3.3417 270	3.7974 983
15	1.9352 8244	2.0789 282	2.3965 582	2.7590 315	3.1721 691	3.6424 825	4.1772 482
16	2.0223 7015	2.1828 746	2.5403 517	2.9521 638	3.4259 426	3.9703 059	4.5949 730
17	2.1133 7681	2.2920 183	2.6927 728	3.1588 152	3.7000 181	4.3276 334	5.0544 703
18	2.2084 7877	2.4066 192	2.8543 392	3.3799 323	3.9960 195	4.7171 204	5.5599 173
19	2.3078 6031	2.5269 502	3.0255 995	3.6165 275	4.3157 011	5.1416 613	6.1159 390
20	2.4117 1402	2.6532 977	3.2071 355	3.8696 845	4.6609 571	5.6044 108	6.7275 000
21	2.5202 4116	2.7859 626	3.3995 636	4.1405 624	5.0338 337	6.1088 077	7.4002 499
22	2.6336 5201	2.9252 607	3.6035 374	4.4304 017	5.4365 404	6.6586 004	8.1402 749
23	2.7521 6635	3.0715 238	3.8197 497	4.7405 299	5.8714 637	7.2578 745	8.9543 024
24	2.8760 1383	3.2250 999	4.0489 346	5.0723 670	6.3411 807	7.9110 832	9.8497 327
25	3.0054 3446	3.3863 549	4.2918 707	5.4274 326	6.8484 752	8.6230 807	10.8347 059
26	3.140¢ 7901	3.5556 727	4.5493 830	5.8073 529	7.3963 532	9.3991 579	11.9181 765
27	3.2820 0956	3.7334 563	4.8223 459	6.2138 676	7.9880 615	10.2450 821	13.1099 942
28	3.4296 9999	3.9201 291	5.1116 867	6.6488 384	8.6271 064	11.1671 395	14.4209 936
29	3.5840 3649	4.1161 356	5.4183 879	7.1142 571	9.3172 749	12.1721 821	15.8630 930
30	3.7453 1813	4.3219 424	5.7434 912	7.6122 550	10.0626 569	13.2676 785	17.4494 023
31	3.9138 5745	4.5380 395	6.0881 006	8.1451 129	10.8676 694	14.4617 695	19.1943 425
32	4.0899 8104	4.7649 415	6.4533 867	8.7152 708	11.7370 830	15.7633 288	21.1137 768
33	4.2740 3018	5.0031 886	6.8405 899	9.3253 398	12.6760 496	17.1820 284	23.2251 544
34	4.4663 6154	5.2533 480	7.2510 253	9.9781 135	13.6901 336	18.7284 109	25.5476 699
35	4.6673 4781	5.5160 154	7.6860 868	10.6765 815	14.7853 443	20.4139 679	28.1024 369
36	4.8773 7846	5.7918 161	8.1472 520	11.4239 422	15.9681 718	22.2512 250	30.9126 805
37	5.0968 6049	6.0814 069	8.6360 871	12.2236 181	17.2456 256	24.2538 353	34.0039 486
38	5.3262 1921	6.3854 773	9.1542 524	13.0792 714	18.6252 756	26.4366 805	37.4043 434
39	5.5658 9908	6.7047 512	9.7035 075	13.9948 204	20.1152 977	28.8159 817	41.1447 778
40	5.8163 6454	7.0399 887	10.2857 179	14.9744 578	21.7245 215	31.4094 200	45.2592 556
41	6.0781 0094	7.3919 882	10.9028 610	16.0226 699	23.4624 832	34.2362 679	49.7851 811
42	6.3516 1548	7.7615 876	11.5570 327	17.1442 568	25.3394 819	37.3175 320	54.7636 992
43	6.6374 3818	8.1496 669	12.2504 546	18.3443 548	27.3666 404	40.6761 098	60.2400 692
44	6.9361 2290	8.5571 503	12.9854 819	19.6284 596	29.5559 717	44.3369 597	66.2640 761
45	7.2482 4843	8.9850 078	13.7646 108	21.0024 518	31.9204 494	48.3272 861	72.8904 837
46	7.5744 1961	9.4342 582	14.5904 875	22.4726 234	34.4740 853	52.6767 419	80.1795 321
47	7.9152 6849	9.9059 711	15.4659 167	24.0457 070	37.2320 122	57.4176 486	88.1974 853
48	8.2714 5557	10.4012 697	16.3938 717	25.7289 065	40.2105 731	62.5852 370	97.0172 338
49	8.6436 7107	10.9213 331	17.3775 040	27.5299 300	43.4274 190	68.2179 083	106.7189 572
50	9.0326 3627	11.4673 998	18.4201 543	29.4570 251	46.9016 125	74.3575 201	117.3908 529

d. $7500 at the end of 20 years at 4% compounded semi-annually.

e. $850 at the end of 12 years at 4% compounded quarterly.

3. If your parents had on the day of your birth placed $1000 to your credit in a savings bank paying 5% compounded semi-annually, how much would it amount to on your twenty-first birthday?

4. Which will amount to the most at the end of one year, $500 at 6% simple interest or $500 at 5% compounded semi-annually? Which will amount to the most at the end of 2 years? At the end of 5 years? At the end of 10 years? At the end of 20 years? Find the difference in each case.

5. A man has $1000 on deposit in a savings bank that pays 4% interest compounded quarterly. He has the chance to invest this money at 5% simple interest. Which will yield the most interest in 10 years? in 12 years? Find the difference in each case.

260. Systematic Saving. In order to make saving a regular habit many people deposit a definite sum in a savings bank at regular intervals. In this connection problems such as the following arise.

Problem. If you were to deposit $100 in a savings bank every six months on January 1 and July 1, for five years, how much would you have at the end of that time if the bank paid 4% compounded semi-annually on the dates mentioned?

Such problems can be solved by means of compound interest tables such as those given on pages 308 and 309. The first $100 deposited remains in the bank and draws interest for 10 periods at the rate of 2% per period. The second $100 draws interest for 9 periods, the next for 8 periods, etc., and the last $100 draws interest for only one period. The total amount at the end of the 10 periods is, therefore, 100 times the sum of the first 10 amounts in the column of the table headed 2%. The student should find this amount to the nearest cent.

COMPOUND INTEREST AMOUNT TABLE.

Showing the amount at compound interest, of periodical investments of $1.00.

At End of What Period	Rate for the Period					
	2%	2½%	3%	3½%	4%	4½%
1	1.020000	1.025000	1.030000	1.035000	1.040000	1.045000
2	2.060400	2.075625	2.090900	2.106225	2.121600	2.137025
3	3.121608	3.152515	3.183627	3.214942	3.246464	3.278191
4	4.204040	4.256328	4.309135	4.362465	4.416322	4.470709
5	5.308120	5.387736	5.468409	5.550152	5.632975	5.716891
6	6.434283	6.547430	6.662462	6.779407	6.898294	7.019151
7	7.582969	7.736115	7.892336	8.051686	8.214226	8.380013
8	8.754628	8.954518	9.159106	9.368495	9.582795	9.802114
9	9.949721	10.203381	10.463879	10.731393	11.006107	11.288209
10	11.168715	11.483466	11.807795	12.141991	12.486351	12.841178
11	12.412089	12.795552	13.192029	13.601961	14.025805	14.464031
12	13.680331	14.140441	14.617790	15.113030	15.626837	16.159913
13	14.973938	15.518952	16.086324	16.676986	17.291911	17.932109
14	16.293416	16.931926	17.598913	18.295680	19.023587	19.784054
15	17.639285	18.380224	19.156881	19.971029	20.824531	21.719336
16	19.012070	19.864730	20.761587	21.705015	22.697512	23.741706
17	20.412312	21.386348	22.414435	23.499691	24.645412	25.855083
18	21.840558	22.946007	24.116868	25.357180	26.671229	28.063562
19	23.297369	24.544657	25.870374	27.279681	28.778078	30.371422
20	24.783317	26.183273	27.676485	29.269470	30.969201	32.783136
	5%	6%	7%	8%	9%	10%
1	1.050000	1.060000	1.070000	1.080000	1.090000	1.100000
2	2.152500	2.183600	2.214900	2.246400	2.278100	2.310000
3	3.310125	3.374616	3.439943	3.506112	3.573129	3.641000
4	4.525631	4.637093	4.750739	4.866601	4.984710	5.105100
5	5.801912	5.975318	6.153290	6.335929	6.523334	6.715610
6	7.142008	7.393837	7.654021	7.902803	8.200434	8.487171
7	8.549108	8.897468	9.259802	9.616627	10.028473	10.435888
8	10.026564	10.491316	10.977988	11.467557	12.021036	12.579476
9	11.577892	12.180795	12.816448	13.466562	14.192929	14.937424
10	13.206787	13.971642	14.783599	15.625487	16.560293	17.531167
11	14.917126	15.869941	16.888451	17.957126	19.140719	20.384283
12	16.712982	17.882137	19.140963	20.475296	21.953384	23.522712
13	18.598631	20.015066	21.550488	23.194920	25.019189	26.974983
14	20.578563	22.275970	24.129022	26.132113	28.360916	30.772481
15	22.657491	24.672528	26.888053	29.304283	32.003398	34.949729
16	24.840366	27.212880	29.840217	32.730225	35.973704	39.544702
17	27.132384	29.905652	32.999032	36.430243	40.301338	44.599173
18	29.539003	32.759992	36.378965	40.426263	45.018458	50.159090
19	32.065954	35.785591	39.995492	44.741964	50.160119	56.275029
20	34.719251	38.992727	43.865177	49.402921	55.764530	63.002529

To avoid the labor of making the additions as above, tables similar to that given on page 311 are used. The table gives directly, without adding, the total amount of periodical deposits of \$1.00, at compound interest, at the end of any number of periods not exceeding 20 and at the rates for the period specified in the table. Thus if \$1 is deposited every six months at 4% per annum compounded semi-annually, at the end of five years it will have remained in the bank 10 periods at 2% for each period and, according to the table, will then amount to \$11.168715. If \$100 is deposited in the same way it will amount to 100 times as much or \$1116.87.

EXERCISES AND PROBLEMS. SYSTEMATIC SAVING.

1. Find the amount at the end of 10 years of \$50 deposited every 6 months at 5% per annum compounded semi-annually. Find this first by using the compound interest table on page 308. As a check use the compound interest amount table on page 311.

2. Find the following, using the compound interest amount table on page 311.

a. The amount at the end of 20 years of \$750 deposited at the beginning of each year at 4½% per annum, compounded annually.

b. The amount at the end of 8 years of \$250 deposited every six months at 5% per annum, compounded semi-annually.

261. Savings Banks. Savings banks usually compute interest on the smallest balance left in the bank for the entire period between two fixed days known as interest days. Some banks compute the interest every month on the smallest monthly balance, others quarterly on the smallest quarterly balance, and some twice a year on the smallest semi-annual balance. Some banks add or compound the interest quarterly, others semi-annually, others annually. Most banks pay no

interest on amounts less than one dollar. The teacher should inform herself as to the customs of local institutions in these matters as they are not uniform.

Problem. The savings bank account of A. B. Smith is shown below. In this case the interest at 4% has been computed monthly on the smallest balance on deposit during the entire month, and has been compounded quarterly on April 1 and July 1.

A. B. Smith

Date		Deposits	Interest	Withdrawals	Balance
1921					
Jan.	12	75.00			75.00
Feb.	7	100.00			175.00
March	23			25.00	150.00
April	1		1.08		151.08
April	15	125.00			276.08
May	3	100.00			376.08
June	7			50.00	326.08
July	1		2.50		328.58

Explanation.

Month	Smallest Balance	Monthly Interest	Quarterly Dividend of Interest
Jan.	0.00	0.00	
Feb.	175.00	.58	
March	150.00	.50	
		——	1.08 (Added to principal on April 1.)
April	151.00*	.50	
May	276.00*	.92	
June	326.00*	1.08	
		——	2.50 (Added to Principal on July 1.)

*No interest on the 8¢.

EXERCISES AND PROBLEMS. SAVINGS ACCOUNTS.

1. Make out A. B. Smith's account as it would be if the bank computed the interest quarterly on the smallest balance on deposit during the entire quarter and added the interest quarterly on April 1 and July 1.

2. Make out the same account as it would be if the bank computed the interest quarterly on the smallest quarterly balance and added it semi-annually on July 1.

3. Make out the same account as it would be if the bank computed the interest semi-annually on the smallest semi-annual balance and added it semi-annually on July 1.

4. Below is given the record of the savings deposits and withdrawals of O. B. Jenkins for one year.

Date	Deposits	Withdrawals
Jan. 7	50.00	
Feb. 10	75.00	
Feb. 27	25.00	
March 14		20.00
March 30	50.00	
May 4	100.00	
June 22		75.00
July 20	50.00	
Aug. 1		150.00
Sept. 9	75.00	
Oct. 12	50.00	
Nov. 18	75.00	
Dec. 17		50.00

Make out the above account as it would be on Jan. 1 in each of the following cases.

a. 5% interest computed monthly on the smallest monthly balance and added quarterly on April 1, July 1, Oct. 1, and Jan. 1.

b. 5% interest computed monthly on the smallest monthly balance and added semi-annually on July 1 and Jan. 1.

c. 5% interest computed quarterly on the smallest quarterly balance and added quarterly.

d. 5% interest computed quarterly on the smallest quarterly balance and added semi-annually.

e. 5% interest computed semi-annually on the smallest semi-annual balance and added semi-annually.

5. Find out the rules of a local savings bank for computing interest and make out the above account according to these rules if you have not already done so.

262. Postal Savings Banks. The government through its post offices now offers a convenient method of depositing savings in the *Postal Savings Banks.* Any person ten years of age or over may open a postal savings account by the deposit of any number of dollars from $1 to $100. Sums less than $1 can not be deposited. Not more than $100 can be deposited in any one calendar month and the total deposits can not exceed $100. Depositors are given certificates of deposit for the amount deposited. Interest is paid on all money that remains on deposit for a full year starting with the first day of the month following the deposit. The money may be withdrawn at any time by the surrender of the certificate.

Provision is made for savings in amounts less than one dollar through 10-cent postal savings stamps. These stamps may be bought separately and ten of them are accepted as a deposit of $1. These stamps bear no interest.

II. The Investment of Savings.

263. Treasury Savings Certificates. The government offers a convenient and safe method of investing savings through the Treasury Savings Certificates. These are on sale at most post offices and are now issued in denominations of $25, $100, and $1,000 maturity value. The $25 certificate costs $20, the

$100 costs $80, and the $1,000 costs $800. As these certificates mature five years from date of purchase, if held to maturity they yield interest at the rate of about 4½ per cent per annum compounded semi-annually. The certificates are redeemable at any time before maturity at the redemption values stated on the backs of the certificates and in that case yield interest at the rate of about 3½ per cent per annum compounded semi-annually.

Postal savings deposits may be exchanged at postal savings offices for Treasury Savings Certificates. When this is done interest is allowed for the deposits withdrawn at the current postal savings rate for each full month up to the first day of the month in which the exchange is made.

Problems. Treasury Savings Certificates.

1. A Treasury Savings Certificate of $25 maturity value and maturing at the end of five years can be bought for $20. Show that this is equivalent to about 4½% per annum compounded semi-annually.

(Hint. Find the amount of $20 at the end of 5 years at 4½% compounded semi-annually, and compare to the maturity value of the certificate).

2. A Treasury Savings Certificate of $25 maturity value is redeemable before maturity according to the following table of values.

	First Year	Second Year	Third Year	Fourth Year	Fifth Year
Issue Price	$20.00				
At End of 6th Month	20.25	20.95	21.70	22.45	23.25
At End of Year	20.60	21.30	22.05	22.85	23.65

At Maturity, 5 years from date of issue, $25.

Show that the certificate yields about 3½% per annum compounded semi-annually, if redeemed before maturity. (Hint. Find the amount of $20 at the end of each half year at 3½% compounded semi-annually and compare to the redemption values).

264. Other Methods of Investing Savings. In addition to the Treasury Savings Certificates now offered by the government there are several other methods of investing accumulated savings with which the teacher and pupils should be familiar. The most important of these are real estate and mortgages, stocks and bonds, and insurance.

II. Real Estate and Mortgages.

265. Real Estate. People often buy houses as an investment and rent them. In making such investments the purchaser should consider the taxes and expenses of keeping up the property and the prevailing rent in his locality for such houses in order to determine whether or not he can get a satisfactory rate of income from his investment. He must also consider whether property values are apt to increase or decrease in that locality in order to make sure that his capital does not depreciate.

266. Mortgages. Persons with money to invest may lend it to individuals who want to build homes but who have not sufficient money, or to farmers to enable them to purchase more land or to improve what they already have. When this is done the borrower gives his note secured by a mortgage on the property. The *mortgage* is in effect a deed in trust, of the property to the debtor. The borrower retains possession and use of the property, but if he fails to pay the interest or principal when it falls due or fails to keep any of the other stipulations of the mortgage, the holder of the mortgage can force the sale of the property and deduct the amount of the loan and interest due from the proceeds, returning the balance, if any, to the bor-

rower. In order to protect the creditor the borrower agrees in the mortgage to pay the taxes and deliver the tax receipts to the lender, to insure the property and make the policies payable to the lender, and to keep the property in good repair.

PROBLEMS. INVESTMENTS IN REAL ESTATE AND MORTGAGES.

1. If you buy a house for $3000 and spend $500 on improvements, at what must you rent it per month to get a net return of 6% on your investment, if the insurance averages $7, the taxes $56 and the repairs $30 a year?

2. At a time when money can be safely invested to bring 6%, would the following house be a good investment? Cost $8000, rent $50 a month, average yearly expense for insurance, taxes and repairs $215. What besides the rate of income should be considered before investing money in a house to rent?

3. Suppose a man bought the house in the preceding problem because he thought it would increase in value. He held it one year and sold it for $8500. How much more did he make than he would have if he had invested the original $8000 at 6%?

4. Mr. L. C. McAdams owns a lot worth $1000 and has $2000 in cash. In order to build a house costing $5000 he wishes to borrow $3000 giving three equal notes payable in 1, 2 and 3 years, respectively, with interest at 5% and a mortgage on the house and lot as security. If you had $3000 in a savings bank drawing 4% interest compounded quarterly, would it pay you to withdraw it and loan it to Mr. McAdams? How much more interest would you receive?

III. STOCKS AND BONDS.

267. Corporations. Formerly business enterprises were usually owned and controlled by individuals or by small groups of individuals acting as partners. At the present time large business enterprises are usually owned and controlled by corporations. The *corporation* is a modified form of partnership

on a large scale, it consists of a number of individuals united into one body which is empowered by the state to act in a certain capacity or to transact some form of business.

268. The Formation of a Corporation. A group of men interested in the formation of a corporation to start or control a business enterprise may apply to the Secretary of State of any state for a *charter*. The application must state the proposed name of the company, the purpose for which it is formed, its location, the proposed amount of capital, the number of shares into which the capital is to be divided and any other provision required by the laws of the state in which the application is being made.

The Secretary of State then grants the organizers the right to sell the capital stock. As soon as the capital stock is sold the organizers meet and elect a board of directors, who elect the necessary officers of the company from among their number. When the record of this is filed with the Secretary of State he issues to the company a charter defining the powers, rights and duties of the corporation.

269. Stock. The capital of the company is not provided by one individual but is divided into a number of parts or shares. If the total authorized capital is \$100,000 it might be divided into 1000 shares of \$100 each nominal or *par value*. These shares are sold for what they will bring which may be more or less than the par value. The price for which the shares actually sell is called the *market value*. The purchaser of a share is given a *certificate of stock*, and becomes a partner in and part owner of the company and as such is entitled to his proportionate share of any profits that may be distributed by the company. The profits that are divided among the stockholders of a company are known as *dividends* and are usually figured as a per cent of the par value of the stock. Thus if a company with a capital stock of \$100,000 decides to distribute profits to the amount of \$3000 among the stockholders it will

declare a dividend of 3% as the profits to be distributed are 3% of the total capital stock. Each stockholder will then receive a dividend of 3% of the par value of the stock which he holds.

It should be noted that the stockholder does not own any part of the property of the company but simply part of the company which is an entirely different thing. The stockholders own the corporation and the corporation owns the property.

In addition to the kind of stock previously discussed, which is known as *common stock,* corporations often issue a second kind known as *preferred stock.* Preferred stock pays a fixed rate of dividend usually from 5% to 7% on the par value. These dividends must be paid from the earnings of the company, if any, before any dividends are paid on the common stock.

270. The Government of a Corporation. Just as the stockholder owns no share in the property of the company, so he does not manage the company's business. The management of the business is in the hands of a board of directors. It is true that these are elected by the holders of the common stock, but as each share of common stock entitles the holder to one vote the person who owns only a few shares usually has no influence in the election of directors, and consequently no influence in the management of the business of the corporation.

271. Bonds. Among the other rights conferred upon a corporation by its charter is that of borrowing money to use in conducting its business. The money to be borrowed is divided into shares usually of $1000 each known as bonds. A bond is a promissory note or a promise to pay a certain amount of money at a fixed time and a fixed rate of interest at stated intervals. The buyer of a bond is therefore loaning his money to the corporation for a stated period and at a fixed rate of interest. The bonds are sold by the company for whatever they will bring. The *market value* of a bond may therefore be more or less than its nominal or *par value.* The market value is

usually given as a certain per cent of the par value. Thus a bond quoted at 104 means 104% of par value, or that a bond of $1000 par value is selling for $1040.

As security for payment of the bonds the corporation usually transfers its property to a trustee, usually a trust company, in trust. The management of the business remains in the hands of the corporation but if they default on the payment of either the interest or principal on the bonds the trustee can in some cases seize the property and operate it in the interests of the bond holders, or he may force the sale of the property in the manner prescribed by law and pay the bondholders from the proceeds. In other words the bonds of a corporation are secured by a mortgage on its property in much the same way as the note of an individual may be secured by a mortgage on property owned by him.

272. Relative Merits of Stocks and Bonds as Investments. *From the above discussion it should be clear that bonds of corporations properly secured by mortgage are much better investments than stock.*

In the first place the interest on these bonds is paid first before the dividends on either the preferred or common stock. Unless the company is bankrupt the bondholders are sure of receiving their interest. If there are no profits the holder of stock receives no dividend. Even if there are profits the holder of common stock can not force the directors to declare a dividend.

In the second place the bond-holder has his principal secured by a mortgage on all of the property of the company. The stockholder has no security for his principal.

Finally the market value of bonds is comparatively staple as they bear a fixed rate of interest and are well secured. This means the bond-holder can sell his bond at any time he may desire and be fairly certain of receiving approximately the amount of his original investment. The market value of stock

on the other hand fluctuates greatly not only as a result of changes in the actual earnings of the corporation but often as a result of manipulation on the part of the directors and others. By withholding dividends even when they have been earned and in various other ways the directors can lower the market value of the stock in order to buy it cheaply from the small stockholder. The holder of stock therefore has no assurance that he will be able to sell it at anywhere near the price he paid.

The only advantage enjoyed by the holder of common stock is the possibility that he may share in large profits if the business engaged in by the company proves to be unusually profitable.

The purchase of the bonds of good companies, properly secured is one of the best forms of *investment;* the purchase of stock is a *speculation.*

Even bonds, however, should not be purchased without proper investigation. The ordinary individual with savings to invest does not usually have either the facilities or the knowledge to enable him to make such an investigation. Bonds should only be purchased, therefore, through and on the advice of a reputable banker or investment-banker. Banks usually charge a brokerage fee of about $\frac{1}{8}\%$ of the par value for buying or selling bonds for customers.

273. Government Bonds. Money is also borrowed by the national and state governments, and by townships, cities, school districts, etc., through the sale of bonds. Such bonds are secured by the taxing power of the divisions of government issuing them. There is probably a larger number of bond-holders and a wider knowledge and understanding of bonds, in this country today than ever before largely as a result of the government's Liberty and Victory Loans.

274. Rate of Income on Bond Investments. The most important consideration to the purchaser of bonds, next to the security of the investment, is the rate of interest he will re-

ceive on his money. Bonds pay a stated rate of interest on the par value but as the investor may pay either more or less than the par value for the bond, it is evident that the actual net return on the money invested is not necessarily the same as the rate of interest that is paid on the par value. Thus, if the investor pays $1070 for a $1000 bond paying 6% on the par value and maturing in 5 years, he receives $60 a year interest which is less than 6% on the money invested. In addition, at the end of 5 years when the bond matures he receives only the par value of $1000 and so loses $70 of the principal invested. It is evident, therefore, that he receives less than 6% net on his investment. On the other hand, if he bought the above bond at $970 he would receive $60 a year interest which is more than 6% of his investment and when the bond matured would receive $1000 which is $30 more than he paid. It is evident, therefore, that he would receive more than 6% net on his investment.

The computation of the rate of income yielded by a bond is quite complex. It is not necessary for the purchaser to be able to perform these calculations as the rate of income yielded by a bond at the market price is usually given in the market quotations or may be obtained from a banker who determines it by means of a table.

The quotations on the various Liberty Loans on February 27, 1922, were as follows:

			Yield
Liberty	3½s	97.00	3.68%
1st	4¼s	97.30	4.43%
2nd	4¼s	97.00	4.47%
3rd	4¼s	97.88	4.63%
4th	4¼s	97.30	4.48%
Victory	4¾s	100.26	4.49%

That is, the Liberty 3½s sold on that day at 97.00% of their par value or at $97 for a $100 bond, and if bought at that price would yield a net return of 4.47% on the investment.

Bond prices are frequently quoted simply in terms of the rate of interest they will yield. The following quotations of this kind are taken from advertisements of bankers and bond houses.

Sweetwater, Texas, Waterworks 5½s, Due 1954, Yielding 5.60%.

Columbia County, Florida, Road 5s, Due 1947, Yielding 5½%.

Western Knitting Mills, 7½%, First Mortgage Bonds, 1930 Maturity, To Yield 5¾%.

Richmond County, N. C., Courthouse 6s, Mature 1932-1935, Price 5.30%.

EXERCISES AND PROBLEMS. STOCKS AND BONDS.

Note.—While a general knowledge of stocks and bonds is highly important to the average individual, a knowledge of the technical arithmetic of these subjects is unnecessary and consequently has no place in the elementary school or in this book. The questions and problems given below are intended merely to illustrate the important points of the general discussion given above.

1. What is the fundamental distinction between stock and a bond? If you own a share of stock in a certain corporation just what is it that you own? If you own a bond of the same company what have you done? If you own a bond of some division of the government what have you done?

2. Why is the buying of good bonds an investment? Why is the buying of stock a speculation? Discuss in detail.

3. Does a bond yield a net return greater or less than the rate of interest paid on the par value, when bought above par? When bought below par?

4. Answer the following questions for each of the four issues of bonds quoted at the top of this page.

a. By whom are the bonds issued?
b. For what purpose is the money to be used?
c. What rate of interest do they pay on the par value?
d. What rate of return do they yield on the money invested if bought at the market price?
e. Are the bonds selling above or below par?

5. Get bond quotations from the newspapers and interpret them in the same way as in the preceding problem.

6. A corporation has outstanding $100,000 of Common Stock, $100,000 of 7% Preferred Stock and $50,000 of 6% bonds. At the end of the year the company has earnings to the amount of $14,000 with which to pay the interest on its bonds, and the dividends on its stock. How much of this will it take to pay the interest on the bonds? How much to pay the dividends on the preferred stock? How much is left to divide among the holders of the common stock? What per cent dividend (based on the the par value) can the company pay on its common stock?

7. If you hold 5 shares of common stock, 5 shares of preferred stock, and 5 bonds of the above company, $100 par value each, how much would you receive on each?

8. Answer the questions of Problem 6 if the corporation has at the end of the year only $10,000 with which to pay interest and dividends.

9. Under the conditions stated in the preceding problem what return would you get on each if you held stock and bonds as stated in Problem 7?

10. Answer the questions of Problem 6 if the corporation has at the end of the year only $3,000 with which to pay interest and dividends.

11. Under the conditions stated in the preceding problem what return would you get on each if you held stock and bonds as stated in Problem 7?

IV. Life Insurance.

275. Definition and Purpose of Life Insurance. "The primary purpose of life insurance is the protection of the family. In the overwhelming majority of cases the subsistence of the family depends upon the current earnings of the husband. He is the breadwinner who has definitely assumed responsibility for the support of those dependent upon him, and his wife and children have a right to look to him for adequate maintenance. His life has a value to the dependent members of the family, and it is this value of one life in relation to another that justifies the existence of life insurance. If a man owns a house he usually allows little time to pass before insuring it. Yet why consider the value of property as more important than the value of the life of the owner when in the great majority of instances the value of the latter to the family exceeds that of the former? As Benjamin Franklin aptly stated: 'A policy of life insurance is the oldest and safest mode of making certain provision for one's family. It is a strange anomaly that men should be careful to insure their houses, their ships, their merchandise, and yet neglect to insure their lives, surely the most important of all to their families, and more subject to loss.' "*

Life insurance may be defined as "that social device for making accumulations to meet uncertain losses through premature death which is carried out through the transfer of the risks of many individuals to one person or a group of persons."†

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*From Huebner's Life Insurance. Used by permission of D. Appleton and Company, publishers.

†From Willett's The Economic Theory of Risks and Insurance. Used by permission of the author and of The Columbia Studies, Columbia University, publishers.

American Table of Mortality.

Completed Age	Number surviving at Each Age	Deaths in Each Year	Death Rate per 1000	Average Future Lifetime — Years	Completed Age	Number surviving at Each Age	Deaths in Each Year	Death Rate per 1000	Average Future Lifetime — Years
10	100,000	749	7.490	48.7	55	64,563	1,199	18.571	17.4
11	99,251	746	7.516	48.1	56	63,364	1,260	19.885	16.7
12	98,505	743	7.543	47.5	57	62,104	1,325	21.335	16.1
13	97,762	740	7.569	46.8	58	60,779	1,394	22.936	15.4
14	97,022	737	7.596	46.2	59	59,385	1,468	24.720	14.7
15	96,285	735	7.634	45.5	60	57,917	1,546	26.693	14.1
16	95,550	732	7.661	44.9	61	56,371	1,628	28.880	13.5
17	94,818	729	7.688	44.2	62	54,743	1,713	31.292	12.9
18	94,089	727	7.727	43.5	63	53,030	1,800	33.943	12.3
19	93,362	725	7.765	42.9	64	51,230	1,889	36.873	11.7
20	92,637	723	7.805	42.2	65	49,341	1,980	40.129	11.1
21	91,914	722	7.855	41.5	66	47,361	2,070	43.707	10.5
22	91,192	721	7.906	40.9	67	45,291	2,158	47.647	10.0
23	90,471	720	7.958	40.2	68	43,133	2,243	52.002	9.5
24	89,751	719	8.011	39.5	69	40,890	2,321	56.762	9.0
25	89,032	718	8.065	38.8	70	38,569	2,391	61.993	8.5
26	88,314	718	8.130	38.1	71	36,178	2,448	67.665	8.0
27	87,596	718	8.197	37.4	72	33,730	2,487	73.733	7.6
28	86,878	718	8.264	36.7	73	31,243	2,505	80.178	7.1
29	86,160	719	8.345	36.0	74	28,738	2,501	87.028	6.7
30	85,441	720	8.427	35.3	75	26,237	2,476	94.371	6.3
31	84,721	721	8.510	34.6	76	23,761	2,431	102.311	5.9
32	84,000	723	8.607	33.9	77	21,330	2,369	111.064	5.5
33	83,277	726	8.718	33.2	78	18,961	2,291	120.827	5.1
34	82,551	729	8.831	32.5	79	16,670	2,196	131.734	4.7
35	81,822	732	8.946	31.8	80	14,474	2,091	144.466	4.4
36	81,090	737	9.089	31.1	81	12,383	1,964	158.605	4.1
37	80,353	742	9.234	30.4	82	10,419	1,816	174.297	3.7
38	79,611	749	9.408	29.6	83	8,603	1,648	191.561	3.4
39	78,862	756	9.586	28.9	84	6,955	1,470	211.359	3.1
40	78,106	765	9.794	28.2	85	5,485	1,292	235.552	2.8
41	77,341	774	10.008	27.5	86	4,193	1,114	265.681	2.5
42	76,567	785	10.252	26.7	87	3,079	933	303.020	2.2
43	75,782	797	10.517	26.0	88	2,146	744	346.692	1.9
44	74,985	812	10.829	25.3	89	1,402	555	395.863	1.7
45	74,173	828	11.163	24.5	90	847	385	454.545	1.4
46	73,345	848	11.562	23.8	91	462	246	532.466	1.2
47	72,497	870	12.000	23.1	92	216	137	634.259	1.0
48	71,627	896	12.509	22.4	93	79	58	734.177	.8
49	70,731	927	13.106	21.6	94	21	18	857.143	.6
50	69,804	962	13.781	20.9	95	3	3	1000.000	.5
51	68,842	1,001	14.541	20.2					
52	67,841	1,044	15.389	19.5					
53	66,797	1,091	16.333	18.8					
54	65,706	1,143	17.396	18.1					

276. Life Insurance Not a Gamble. If an insurance company were to insure the life of one individual for $1000 it would be a gamble pure and simple. The company upon the receipt of a sum of money, say $20, would enter into a contract agreeing to pay $1000 if the insured dies within a year. That is, the company would be betting the individual that he will not die within that time, if he does the company must pay $1000, if he survives the company pays nothing and gains the $20 received in payment for the insurance. If, on the other hand, the company enters into 100,000 such contracts the element of chance and of gambling disappears. From statistics based on a large number of individuals tables have been compiled showing the number of deaths to be expected out of each 1000 persons at different ages. Such a table is given on the preceding page.

Experience has shown that the actual number of deaths in a group of 100,000 individuals of similar physical conditions will not vary by 2% from the figures given in the table, while if 500,000 individuals are taken the actual number of deaths will vary only a small fraction of 1% from the number given. Several American companies now have insured more than 500,000 lives each. It is evident that if a sufficiently large number of individuals are insured the company can predict very accurately the number of claims it will have to pay at the end of the year and can charge an amount for the insurance sufficient to pay all claims.

277. Life Insurance a Duty. It has been shown that life insurance is not a gamble but rather a mathematical certainty as far as the company is concerned. Neither is life insurance a gamble on the part of the individual taking the insurance. To insure one's life is not a gamble, but not to insure it is.

Any person who has a family or others dependent on him for support and who does not insure his life is gambling. He is taking the chance that he will live long enough to accumulate property and savings sufficient to provide for his dependents after his death, he is gambling with the future happiness and comfort of those nearest and dearest to him. No man has the right to take such chances. Through life insurance he can change the uncertainty of the financial condition of his family after his death into the certainty of an assured income.

278. Definitions. The contract entered into between the company and the individual insured is called a *policy*. The amount paid by the insurer for the insurance is called the *permium*.

279. Ordinary Life Policies. Many different kinds of life insurance policies are issued by American companies. The simplest of those commonly issued is the *ordinary life policy*. The premiums on this type of policy are usually paid annually and are paid throughout the life of the insured. The amount of the policy is paid upon the death of the insured to his heirs or to some party designated by him as the *beneficiary*.

As the death rate increases with the age of the insured it is evident that the actual cost of insurance increases from year to year. Thus according to the American Table of Mortality, if a company insures 85,441 individuals 30 years of age for $1000 each, at the end of the year it can expect to be called upon to pay death claims to 720 individuals amounting to $720,000. If this is divided evenly among the 85,441 individuals insured the cost of the insurance to each would be $720,000÷85,441=$8.43. The second year the company would have to pay 721 claims amounting to $721,000 and at the beginning of the year there would be 84,721 individuals living to share the expense. The cost to each would therefore be $721,000÷84,721=$8.51. At the end of the fortieth year the company must pay 2,321 claims amounting to $2,321,000 and there are

only 40,890 individuals of the original number insured still alive to share this expense. The cost to each is thus $2,321,000 \div 40,890 = \$56.76$.

It was shown in the preceding paragraph that if the insured simply paid the cost of his insurance from year to year he would have to pay an increasing premium as he becomes older and his earning capacity becomes smaller. This method of paying premiums has never been popular so it is customary to charge a uniform yearly permium. This uniform yearly premium must obviously be greater than the actual cost of the insurance in the early years of the policy, but is less than the actual cost in the later years. As a result of this method of paying premiums the insured for a number of years is paying into the company an amount over and above the cost of his insurance for these years. This money, known as the *reserve,* belongs to the insured and is invested for him by the company. The uniform yearly premium is fixed at such a figure that the excess over the cost of the insurance received by the company in the early years of the policy is sufficient, when invested so as to bring interest at 3% or $3\frac{1}{2}\%$, to accumulate a fund sufficient to make up the difference between the uniform yearly premiums received and the actual cost of the insurance in the later years of the policy. A sufficient amount is also added to the yearly premiums to pay the expenses of the company.

If the policy is kept in force until the death of the insured the reserve build up in the earlier years of the policy is used up in paying the increased cost of the insurance and the amount of the policy at death. If, however, the insured allows the policy to lapse before his death he is entitled to the reserve he has built up by paying more than the actual cost of his insurance from year to year. The amount of this reserve, known as the *cash surrender value* of the policy, will be paid by the company to the insured upon the surrender of the policy.

280. Limited Payment Policies. On the ordinary life policy the premiums are paid during the entire life of the insured. Owing to the fact that earning capacity usually decreases with old age, most people prefer to pay larger premiums for a definite number of years in order to avoid the payment of premiums in their old age. Insurance companies issue policies that may be paid in full by a certain number of equal yearly payments. Such policies are known as *limited payment* policies. The most popular of these are the 10-payment and 20-payment policies. Although the insured stops paying premiums at a given time, it should be understood that the insurance is not paid by the company until the death of the insured.

The annual premiums on the 10-payment life must be sufficient to pay the cost of the insurance for the ten years during which payments are made and in addition to create a reserve fund which at 3% or 3½% interest will provide an amount sufficient to pay the cost of the insurance each year after payments have ceased. The premiums on such a policy must evidently be much higher than on the ordinary life, and the reserve is built up much more rapidly. The cash surrender value of such a policy is, therefore, greater than for the ordinary life.

281. Endowment Insurance. Some people prefer a form of policy that will enable them to accumulate a sum of money for their own use and at the same time will afford protection to dependents in the event of the death of the insured before the policy matures. Companies issue various forms of *endowment* insurance policies to meet this need. The commonest forms are the 10-year and 20-year endowments.

If a man takes out a 10-year endowment policy for $1000, he pays premiums for 10 years and at the end of the 10 years receives $1000. The uniform yearly premium on such a policy must evidently be sufficient to pay the death claims of those dying each year before the 10 years is up and to create a reserve

which at 3% or 3½% interest will accumulate a fund large enough to pay all the remaining policies in full at the end of the 10 years. The premiums on such a policy are therefore, higher than on either of the types previously considered and the reserve is built up more rapidly.

282. Life Insurance as a Means of Saving and Investment. Although the primary purpose of life insurance is the protection of one's family most forms, through the creation of a reserve which is invested by the company and pays interest at 3% or 3½%, afford in addition to this protection a means of accumulating and investing savings. For a given yearly premium the ordinary life gives the greatest protection and the least accumulation of savings, the limited payment life gives less protection and greater accumulation of savings and endowment insurance gives the least protection and the greatest accumulation of savings. This is owing to the fact that the reserve is built up more rapidly in the limited payment than in the ordinary life and more rapidly still in the endowment insurance. According to the rates of one company an annual premium of $200 will, at the age of 30, pay for a $10,000 ordinary life policy, for a $7,000 twenty-payment life and for a $4,800 twenty-year endowment. If the holder dies before the end of twenty years his beneficiary would receive $10,000 from the ordinary life, only $7,000 from the 20-payment, and only $4,800 from the 20-year endowment. On the other hand if the insured lives until the end of the 20 years the cash surrender value of the ordinary policy is only $2,500 while that of the 20-payment is $3,500, and that of the endowment is $4,800.

One of the commonest arguments against life insurance as a method of insuring the future support of dependents is the argument that if the money paid for the insurance is deposited in a savings bank at compound interest, it will eventually amount to a larger sum than if put into life insurance. While

there is an element of truth in this argument it must be remembered that it takes a considerable time to accumulate a given sum, say $1000, by the deposit of small savings, whereas the beneficiaries of a $1000 life insurance policy are assured of the full amount, in case of the death of the insured, *as soon as the first payment is made.* To depend upon savings alone to insure an income to one's family is again gambling with their future happiness. Every one should save and invest their savings but in addition any one having dependents should carry sufficient insurance to assure them an income in case of death before savings have reached a sufficient amount.

ANNUAL PREMIUM OF $1000 OF INSURANCE

Age	Ordinary Life	10 Payment Life	20 Payment Life	10 Year Endowment	20 Year Endowment	Age
21	$18.40	$46.30	$28.25	$101.78	$48.08	21
22	18.80	47.00	28.69	102.01	48.17	22
23	19.23	47.73	29.15	102.24	48.25	23
24	19.67	48.47	29.63	102.49	48.35	24
25	20.14	49.24	30.12	102.73	48.46	25
26	20.63	50.04	30.63	102.85	48.58	26
27	21.15	50.87	31.16	102.93	48.69	27
28	21.69	51.72	31.71	103.03	48.83	28
29	22.26	52.61	32.28	103.14	48.96	29
30	22.85	53.52	32.87	103.25	49.12	30
31	23.48	54.46	33.49	103.37	49.28	31
32	24.14	55.44	34.13	103.50	49.46	32
33	24.84	56.45	34.80	103.65	49.66	33
34	25.58	57.50	35.49	103.79	49.87	34
35	26.35	58.58	36.22	103.95	50.11	35

The taking out of insurance also has the effect of encouraging, almost forcing, the saving of money. Most men will keep up the payments on insurance if it is at all possible to do so, because of the loss involved in dropping the insurance

and also because it soon becomes a habit. The same men, however, usually could not trust themselves to deposit the same amount of money regularly in a savings bank. Further it is easier to get the money out of the bank than it is to get it back after paying it for insurance so the temptation to withdraw and spend it is greater.

Problems. Life Insurance.

Note—The mathematics of life insurance is very complex and the calculations are performed by *actuaries* in the employment of the insurance companies. Everyone, however, should have a general knowledge of life insurance and the problems given below are intended to emphasize some of the more important questions with which all should be familiar.

1. A young man, 24 years of age, with a wife and two children totally dependant on his earnings for support, takes out a $4000 twenty-payment life policy, at an annual premium of $29.63 per $1000. What is his total premium?

2. If the man in the preceding problem dies at the end of two years (before the third premium is paid) how much more will his family receive from the insurance than they would have realized if he had deposited the yearly premiums in a savings bank at 4% interest compounded annually?

3. If the man described in the preceding problems died at the end of 10 years (before the eleventh premium was paid) which would yield the greater amount to his family, the insurance, or the amount of the premiums deposited yearly in the savings bank as in the preceding problem? What is the difference? (Suggestion: Use the compound interest amount table on page 311, to determine the amount of the savings bank deposits).

4. Compare the insurance to the amount of the savings bank deposits if the man of the preceding problems died at the end of the twentieth year.

5. Which is the best method of providing an income for dependents, life insurance, or the regular investment of savings? Why?

6. A young man 21 years of age, clerking in a store, wishes to save money so as to accumulate sufficient capital to enable him to buy an interest in the store by the time he is 31 years old. He can save about $200 a year. He finds that he can get a $1000 ten-year endowment policy at an annual premium of $101.78. Would it be better for him to take out a $2000 policy of this kind or to deposit the amount of the annual premiums in a savings bank each year at 4% compounded annually? What is the difference?

7. Can you think of any reason why the man in the preceding problem might prefer to take out the insurance policy?

8. Describe each of the following types of policy.

a. Ordinary Life.
b. Limited Payment Life.
c. Endowment.

9. What kind of policy would you advise each of the following men to take? (Make use of the table of annual premiums on page 333.

a. A man 35 years of age, with an assured income of $2000 a year as long as he lives, but which ceases at his death. He has a wife and two minor children dependent on his income for support. He can invest about $200 a year in insurance.

b. A young man 21 years of age, with an income of $1200 a year. He has a mother 51 years old dependent on his income. He can invest about $125 a year in insurance. He wants the insurance to protect his mother in case he should die first, but expects to marry later and to continue the insurance after his mother's death for the protection of his wife.

c. A man 35 years of age with an income of $4000 a year from his own labors. His income is apt to decrease after 45 years of age. He has a dependent family. He can invest about $400 a year in insurance.

d. A young man 20 years of age with an income of $1000 a year. No dependents. He desires insurance as a means of forcing himself to save. He can invest about $100 a year in insurance.

e. A man 25 years of age, with an income of $2000 a year, but apt to decrease after he is 45 years old. Dependent family. He can invest about $150 a year in insurance.

10. The instructor and students should suggest other cases, differing from those suggested in the preceding problem, and consider the most desirable types of policy for each case.

V. Fire Insurance.

283. Definition and Purpose of Fire Insurance. The primary purpose of fire insurance is the protection of capital invested in buildings or other property that might be destroyed by fire. Fire insurance can not prevent the occurrence of the fire but it does provide a method of reimbursing the owner of the property for his loss. Fire insurance has been defined as "that social device for making accumulations to meet uncertain losses of capital through the transfer of the risks of many individuals to one person or a group of persons."*

284. Fire Insurance Not a Gamble. If an insurance company were to insure one house valued at say $10,000 against damage or loss by fire, it would be a gamble. Upon receipt of a sum of money, say $25, the company would agree to reimburse the owner for any loss due to fire within one year.

*From Willett's The Economic Theory of Risks and Insurance. Used by permission of the author and of The Columbia Studies, Columbia University, publishers.

If no fire occurred the company would profit to the extent of $25 less expenses. In the case of fire the company might have to pay any amount up to $10,000. On the other hand, if the company were to insure 10,000 such houses, it would be possible for it to predict on the basis of past experience what per cent of loss to expect and to charge an amount sufficient to enable it to pay all losses, expenses, and a reasonable profit. By combining a large number of risks the element of chance is eliminated.

285. Definition of Terms. The contract entered into between the company and the owner of the property is called the *policy*. The amount paid for the insurance is called the *premium*.

286. Social Importance of Fire Insurance. The most important service of fire insurance to society is in changing large uncertain losses into small certain losses by sharing them among a large number of individuals. In addition to this it makes possible the modern credit system which is the basis of all commerce and industry. If it were not for fire insurance it would be impossible to borrow money on property as security, as the risk of loss through fire would be too great.

287. Terms of Policy. The form and terms of policy differ somewhat in the various states, but are alike in most essentials. Among the most important clauses in the policy are, (1) the clause or clauses defining the risk assumed by the company, (2) the clause concerning other insurance, and (3) the co-insurance clause.

288. Risks Assumed. The usual form of contract insures against all direct loss or damage by fire. No matter what the value of the property as stated in the policy may be the company is never liable beyond the actual cash value of the property at the time the loss occurs. Property worth $5000 at the

time it was insured and insured for that amount might have depreciated in value to $4000 at the time of the fire, in which case the company would be liable only to the extent of $4000. This clause is necessary as otherwise the insured would be in a position to benefit from a fire and as a result might become careless or even be tempted to incendiarism.

289. Other Insurance Clause. The standard form of policy prohibits the taking out of other insurance on the same property except with the knowledge and sanction of the company. The purpose of this clause is again to prevent over-insurance and the resulting increase in losses due to carelessness and incendiarism.

Where several policies have been written on the same property, with the consent of the companies, each company is liable for the same fractional part of the loss that the amount of its policy is of the total amount of insurance. Suppose the owner of property valued at $30,000 has it insured for a total of $24,000 in three companies, for $10,000 in company A, for $8,000 in company B, and for $6,000 in company C. If a fire loss of $8,000 occurs, company A will be liable for $\frac{10}{24}$ of $8,000=$3,333.33, company B for $\frac{8}{24}$ of $8,000=$2,666.66, and company C for $\frac{6}{24}$ of $8,000=$2,000.

290. Co-Insurance. Most fires result in only a partial loss. As a result some property owners might be tempted to insure their property for only a small amount and run the risk of larger loss. Such a practice, as it does not reduce the total fire loss of the country, must necessarily increase premium rates. As a result the owner who must carry full insurance has to pay a higher rate as he is really helping to pay for the partial losses of those who are only partly insuring their property. To avoid this injustice and to force each property owner to pay his just proportion of the expense of fire insurance many policies contain a *co-insurance,* or *reduced average* clause reading as follows:

"This company shall not be liable for a greater proportion of any loss or damage to property described herein than the sum hereby insured bears to...per centum of the actual cash value of said property at the time such loss shall happen."

It is the usual practice of companies under this clause to require the property owner to insure his property for 80% of its value or failing in this to assume a proportionate share of the risk and loss. Under some circumstances the insured is required to insure his property for its full value.

If a man owns property valued at $15,000 he would be required under the 80% co-insurance clause to insure it for 80% of $15,000 or for $12,000. If he does this the company agrees to pay all losses in full up to $12,000. If, however, he insures it for only $6,000 or for one-half of the required amount, the company would pay only one-half of any loss and its maximum liability would be $6,000 or the face value of the policy. In other words, under these conditions the man is taking and paying for only half insurance and is assuming half of the risk himself. If instead of $6,000 the owner of the above property takes out $10,000 insurance he is insuring for $1\frac{1}{2}$ or $\frac{5}{6}$ of the required amount and the company will pay $\frac{5}{6}$ of all losses. The maximum liability of the company in this case would be $10,000 the face value of the policy.

Under the 100% co-insurance clause the owner of property valued at $15,000 would be required to insure it for its full value. If he took out only $5,000 insurance he would be paying for only one-third insurance and assuming two-thirds of the risk himself. Under these conditions the company would pay only one-third of any loss, its maximum liability being $5,000 the face value of the poilcy.

Problems. Fire Insurance.

1. The fire insurance rates on a building depend upon a number of different factors, including (a) the construction, (b) the occupancy, (c) the exposure, and (d) the facilities for

extinguishing fires. Discuss how each of these would affect the rate.

2. A building valued at \$10,500 is to be insured for \$9000. The premium is 24¢ per \$100 for one year or 60¢ per \$100 for a 3-year term. Find the yearly premium if insured for one year only. What is the cost per year if it is insured for a three-year period? How much can be saved per year by insuring for the three years?

3. A building valued at \$8,000 when insured, and insured for that amount, is totally destroyed by fire. At the time of the fire the building was worth only \$7,000. How much insurance would the owner collect?

4. A building worth \$25,000 is insured in three different companies for \$5,000, \$7,000, and \$8,000. It was damaged by fire to the extent of \$5,000. How much will each company have to pay if there is no co-insurance clause?

5. A building worth \$10,000 is insured for only \$2,000, no co-insurance clause. The building is damaged by fire to the extent of \$1,000. How much insurance will the owner receive? How much will he receive if the damage is \$2000? If the building is totally destroyed?

6. For how much must a building worth \$12,000 be insured to satisfy the 80% co-insurance clause? To satisfy the 100% co-insurance clause?

7. A building worth \$25,000 is insured for \$15,000. If the policy contains an 80% co-insurance clause what part of the risk is the owner assuming?

8. A building worth \$18,000 is insured for \$12,000, the policy containing a 100% co-insurance clause. What part of the risk is the owner assuming?

9. Property worth \$30,000 is insured for \$10,000 no co-insurance. If the property is damaged by fire to the extent of \$8,000 how much insurance must the company pay?

10. If the policy in the preceding problem had contained an 80% co-insurance clause, how much would the company have had to pay?

11. If the policy in Problem 9 had contained a 100% co-insurance clause, how much would the company have had to pay?

12. Why can companies give a reduced rate when making a co-insurance clause part of the contract?

13. Property worth \$42,000 is insured for \$14,000 with an 80% co-insurance clause. In the case of total loss from fire, how much must the company pay?

CHAPTER XVI.

THE ARITHMETIC OF CIVIC LIFE

291. Purposes of Taxation. Probably no application of arithmetic is of more universal interest than the subject of taxes. Too many citizens take a one sided view of this subject. They think only of the money they pay and do not stop to think of what they are getting for their money. The schools can do much to bring about a better attitude on the part of the general public. To do this the subject of taxes should be approached through a study of what the various subdivisions of the government do for the individual citizen. Through such a study the pupils see that towns and cities give police and fire protection; maintain schools, streets and sidewalks; light the streets at night, etc. Counties build and maintain roads and bridges, provide courts, take care of the poor, etc. States support schools and universities, help build and maintain roads, provide asylums and penitentiaries, etc. Finally the national government maintains an army and a navy for the protection of its people, provides courts of justice, provides the post office system, etc.

This study of what the goverment does for the individual naturally leads to the question of how the government gets the money to carry on these enterprises.

292. Kinds of Taxes. The money to pay the expenses of state and local governments is raised largely by taxes on real estate and personal property. The money to meet the expenses of the national government is raised largely by customs duties on imports, internal revenue taxes such as that on tobacco, and the income tax.

293. State and Local Taxes. The amount of the tax paid by any individual depends upon two things, (1) the value of the property he owns, and (2) the tax rate.

294. Determining the Value of the Property. In most states the value of real estate and personal property, for the purpose of taxation, is determined by the tax assessor. This value is known as the *assessed value* and is usually only a fractional part of the real value. As the method of determining the value of the property to be taxed varies in different states the teacher and pupils should make a study of the method used in their state.

295. Determining the Tax Rate. The tax rate is determined by comparing the total amount to be raised by taxation to the total assessed value of the property to be taxed. The tax rate is expressed as a per cent, as so many mills on each dollar of assessed valuation, or as so many dollars and cents on each one hundred dollars of assessed valuation. A tax rate of 2.04% is equivalent to one of 20.4 mills (on each dollar), or $2.04 (on each hundred dollars).

The rate of taxation and the distribution of the tax are usually published in the local newspaper, or this information can be obtained by the teacher from local officials. Most of the problems solved by the pupils should be based on the local rate.

Problems. State and Local Taxes.

1. The total assessed valuation of real and personal property in the state of New York for the fiscal year ending June 30, 1921, was $14,850,989,607. The amount to be raised by direct taxation was $22,340,344. Find the tax rate for state purposes. Express the rate in three ways.

2. The total assessed valuation of real and personal property in Wood County, Ohio, in 1921, was $99,667,790. The amount to be raised by taxation for county purposes was $411,130. Find the tax rate for county purposes. Express in three ways.

3. From local officials find the total assessed valuation of real and personal property in your county and the amount to be raised by taxation. Find the tax rate for county purposes.

4. The total tax rate for all purposes in Toledo, Ohio, in 1921, was 20.4 mills on the dollar. Find the amount of real and personal tax to be paid by a man owning real estate assessed at $7,000 and personal property assessed at $1,500.

5. From local officials find the total tax rate in your town. What tax would you have to pay on property assessed at $8,000.

INCOME TAX RATES FOR 1922 AND THEREAFTER.

Net Income	Rate of Normal Tax Percent	Rate of Surtax Percent	Amount Subject to Surtax Between Change of Rates
$3,000	4		
4,000	4		
5,000	4		
6,000	4		
8,000	8	1	$6,000 to 8,000
10,000	8	1	8,000 to 10,000
12,000	8	2	10,000 to 12,000
14,000	8	3	12,000 to 14,000
16,000	8	4	14,000 to 16,000
18,000	8	5	16,000 to 18,000
20,000	8	6	18,000 to 20,000
22,000	8	8	20,000 to 22,000
24,000	8	9	22,000 to 24,000
26,000	8	10	24,000 to 26,000
28,000	8	11	26,000 to 28,000
30,000	8	12	28,000 to 30,000

296. The Income Tax. Of the taxes levied by the national government only the income tax is of sufficient general interest to justify consideration here. The United States Revenue Act of 1921 provides for a normal tax and a surtax on incomes. The table on preceding page shows the rates on net incomes of $30,000 and less. Net income as used in the table means gross income less (1) the necessary expenses of carrying on a trade or business, (2) interest paid on indebtedness, (3) losses in any transaction entered into for profit, and (4) debts ascertained to be worthless and charged off within the taxable year.

In computing the normal tax the following personal exemptions are allowed $1,000 to a single person; $2500 to the head of a family or married person living with husband or wife, unless the net income is in excess of $5,000, in which case the personal exemption is only $2,000; $400 for each person (other than husband or wife) dependent upon and receiving his chief support from the taxpayer if such dependent is under eighteen years of age or is incapable of self-support.

Problem 1. Find the amount of income tax on a net income of $4,000, married, no dependents.

Solution.

Net Income	Amount Subject to tax	Rate	Amount of tax
All up to $2500	Exempt		
$2500 to $4000	$1500	4%	$60.
Total Tax			$60.

Problem 2. Find the amount of income tax on a net income of $16,000, married, no dependents.

Solution.

Normal Tax.

Net Income	Amount Subject to Tax	Rate	Amount of Normal Tax
All up to $2,000	Exempt		
$2,000 to 6,000	$ 4,000	4%	$160.
6,000 to 16,000	10,000	8%	800.
Total Normal Tax			$960.

Surtax.

Net Income	Amount Subject to surtax	Rate	Amount of Surtax
All up to $6,000	None		
$6,000 to 10,000	$4000	1%	$40.
10,000 to 12,000	2000	2%	40.
12,000 to 14,000	2000	3%	60.
14,000 to 16,000	2000	4%	80.
Total Surtax			$ 220.
Total Tax			1,180.

Problem 3. Find the amount of income tax on a net income of $12,000, married, two dependents.

Solution.

Normal Tax.

Net Income	Amount Subject to Tax	Rate	Amount of Normal Tax
All up to $3300	Exempt		
$3300 to $6,000	$2,700	4%	$108.
6,000 to 12,000	6,000	8%	480.
Total Normal Tax			$588.

Surtax.

Net Income	Amount Subject to tax	Rate	Amount of Surtax
All up to $6,000	None		
$6,000 to $10,000	$4,000	1%	$40.
10,000 to 12,000	2,000	2%	40.
Total Surtax			$ 80.
Total Tax			$668.

PROBLEMS. INCOME TAX.

1. Find the amount of income tax on each of the following.

Net Income	Married or Single	Number of dependents
a. $ 3,000	single	none
b. 4,000	married	one
c. 10,000	married	three
d. 10,000	single	one
e. 25,000	married	two
f. 30,000	married	one

CHAPTER XVII.

THE ARITHMETIC OF BUSINESS.

I. Introduction.

297. General Discussion. Although the elementary school can not give a comprehensive course in business arithmetic it must attempt to give its pupils some insight into the important problems in this field. The only training that the majority of the smaller retail merchants of this country receive is that received in the elementary schools.

A merchant must sell his goods for an amount sufficient to cover the cost of the goods and the expenses of running his business and in addition to give him a reasonable profit. The three most important problems of the merchant are, therefore, the determination of costs, the figuring of expenses and the fixing of selling prices so as to afford a profit.

298. Definitions. The original cost of the goods to the merchant is called the *net cost* or *prime cost.* All expenses incurred in getting the goods to the store and ready to sell are known as *buying expenses* and include such items as freight, drayage, insurance and storage while in transit, etc. The general expenses incidental to running the business such as rent, light, heat, advertising, bookkeeping, taxes and depreciation are called *overhead charges* or *overhead.* The expenses incurred in selling the goods such as commissions or salaries of clerks and salesmen are called *selling expenses.* The buying expenses, overhead and selling expenses combined are called the *cost of doing business* or *total expenses.* The prime cost plus the cost of doing business is the *total cost.* The difference between the net cost and the selling price is the *gross profit.* The gross profit less the cost of doing business is the *net profit.*

II. Buying Goods.

299. Trade Discounts. The retail merchant buys his goods of a wholesaler or in some instances directly from the manufacturer. The purchase is usually made with the assistance of a catalog in which are given descriptions and illustrations of the various articles.

The price given in the catalog is called the *list price.* This is not usually the wholesale price or the price which the retailer must pay the wholesaler for the article, but is at least as high as the retail price, or the price at which the retailer sells the article to his customers. This is done to enable the retailer to sell at the list price or less as the catalogs are often shown to his customers so that they can order articles that he does not have in stock. To enable the retailer to find the price he must pay for an article he is given a confidential *discount sheet,* on which are given the discounts allowed the retailer on the various articles in the catalog. These discounts are stated as per cents of the list price and are to be deducted from the list price. To illustrate, if an article is listed in the catalog at \$35 and the retailer is given a discount of 20% he will get a discount of 20% of \$35 or of \$7 and will have to pay for the article \$35—\$7=\$28. The per cent of the list price to be deducted is called the *rate of discount,* the amount to be deducted is the *amount of the discount* or simply the *discount* and the price paid by the retailer or the amount left after subtracting the discount from the list price is the *net price* or *invoice price.* This invoice price is the prime cost of the goods to the retailer.

Discounts such as described in the preceding paragraph, allowed the retailer by the wholesaler or manufacturer, are known as *trade discounts.*

Some catalogs do not give prices. In such cases the retailer is furnished with a confidential price list. As this price list is never shown to customers the prices given are those

which must be paid by the retailer and are not subejct to discount. If the catalog is of such a character that it need never be shown to the general public the prices the retailer must pay are printed in the catalog and no discount is given.

300. Series of Trade Discounts. Changes in prices after a catalog has been published are accomplished by changing the rate of discount. If there is an increase in price the original discount is withdrawn and a smaller one substituted. Thus if an article is listed at \$80 with a discount of 25% it has been selling at a net price of \$80—\$20=\$60. If the wholesaler must increase the price so as to receive a net price of \$70 the new discount will be \$80—\$70=\$10 which is $\frac{1}{8}$ or 12½% of the list price.

If there is a decrease in price either of two things may be done. The original discount may be withdrawn and a larger one given or the original discount may be left and a second discount added. Thus an article listed at \$200 with a discount of 25% has been bringing a net price of \$200—\$50=\$150. If the wholesaler can now sell it for \$140 the new discount will be \$200—\$140=\$60 which is $\frac{3}{10}$ or 30% of the list price.

Sometimes the net price is decreased by leaving the original discount and adding a second which is based on the original net price. Thus an article listed at \$150 with a discount of 20% has been bringing a net price of \$150—\$30=\$120. If the wholesaler can now sell it for \$110 he can give a second discount of \$10 which is $\frac{1}{12}$ or 8⅓% of \$120 the original net price, so he will now give discounts of 20% and 8⅓% on the article. When two or more trade discounts are given it is called a *series of discounts.* The first discount is always based on the list price, the second on the amount left after the first is deducted, etc.

Series of discounts also arise in other ways. Sometimes a regular trade discount is allowed and an additional discount is allowed on orders over a certain amount. This is done to

encourage large orders as they are less expensive in proportion to handle than smaller orders. A manufacturer selling to wholesalers and jobbers and also directly to retailers often gives two discounts, one to retailers and an additional one to jobbers.

301. Invoices. When the goods purchased by a retailer are shipped they are usually accompanied by an itemized statement showing goods bought, amounts, list prices, and trade discounts. Such a statement is known as on *invoice.* The goods received are compared with the invoice to detect any shortage or mistakes in the order.

302. Cash Discounts. The invoice also usually states the terms on which the goods are sold, as 2% 10 *days; net* 30 *days.* This means that payment of the net amount is due in 30 days, but that a discount of 2% from the net amount is allowed for payment within 10 days. A discount given for payment within a certain time is called a *cash discount.*

The A. B. Moon Company,
Toledo, Ohio.

February 20, 1922.

SOLD TO *The Reynolds Furniture Co.,*
Bowling Green, Ohio.

Terms: 2% 10 days; net 30 days.

2	Mahogany Chairs 2175	30.00	60	00		
2	Mahogany Rockers 2176	31.00	62	00		
1	Davenport 1758	125.00	125	00		
1	Dining Room Suite 2389	350.00	350	00		
			597	00		
		Less 33⅓ %	199	00		
					398	00

Problems. Trade and Cash Discounts.

1. A hardware dealer wishes to order a certain kind of stove for a customer. He finds from the catalog of the manufacturer that the list price is $62.50. From the confidential discount sheet furnished him by the manufacturer he finds that a discount of 30% of the list price is given to retail dealers. What will he have to pay for the stove?

2. If the Reynolds Furniture Company pays for the bill of goods covered by the invoice on page 350 by February 30, what will they have to pay? If they do not pay until after that date what must they pay?

3. A retail merchant buys goods from a manufacturer amounting to $395. He receives discounts of 20% and 10%. Terms: 10 days 2%, 30 days net. Find the amount of the bill if cash is paid.

4. Find the cash price in each of the following.

	Total List Price	Discounts	Terms
a.	$872.50	33⅓%	Net cash
b.	584.30	25%, 10%	1% 15 days; net 30 days
c.	79.50	40%	2% cash; net 60 days
d.	187.40	20%, 10%, 10%	30 days net
e.	231.70	33⅓%, 20%	2% 10 days; net 30 days

5. Make out an invoice for the following bill of goods bought of Underwood and Sons, Chicago, Illinois, by O. N. Norris, Toledo, Ohio. Terms: 10 days 2%, 30 days net. 750 feet of 2″ iron pipe at 42¢, 41 elbows (2″) at 25¢, 500 feet of 3″ iron pipe at 59¢, 20 elbows (3″) at 35¢, all less 20% and 10%.

6. How much does O. N. Norris have to pay for the above bill of goods if he pays cash?

7. A jobber buys goods amounting to $2384.42 from a manufacturer who allows 33⅓% discount to retail dealers and an additional discount of 20% to jobbers. Terms 2% cash. Find the cash amount of the bill.

8. A publisher of school books lists all books so as to allow dealers and schools a discount of 20%. At what must this publisher list a book for which he wishes to receive a net price of $1.25?

9. A wholesaler lists an article in his catalog at $50. If he wishes to receive a net price of $35 what discount can he allow?

10. A manufacturer allows a uniform trade discount of 33⅓% on all articles in his catalog. At what must he list an article in order to receive a net price of $35?

11. A steel tape was listed in a jobbers catalog at $8.00 with a discount of 20%. Owing to an increase in the cost of the tape during the war the jobber found that he must receive a net price of $7.00. What did he do?

12. Owing to a decrease in the cost of raw materials a manufacturer wishes to make a net price of $60, on an article listed in his catalog at $100 and on which he has been allowing a discount of 25%. What can he do? Answer in two ways.

13.

Net amount of bill	Terms	Rate of interest merchant must pay on borrowed money.
a. $756.50	2% cash, net 60 days	6%
b. 917.40	1% 15 days, net 30 days	7%
c. 349.07	2% 10 days, net 30 days	7%

If it is necessary to do so, merchants often borrow money so as to be able to take advantage of cash discounts. They can

usually save money by doing so and in addition the reputation of always paying cash is a distinct asset as it improves their credit standing. Find what the merchant can save by borrowing the money necessary to take advantage of the cash discount in each of the cases given in the table.

III. Figuring Expenses.

303. Methods of Figuring Expenses. As it is not possible to apportion expenses among the various items sold it is customary to express them as a per cent either of the net cost of goods sold or of the total sales. Suppose that in one year a merchant's sales amount to \$10,000, the goods sold cost \$6000 and the total cost of doing business is \$2000. Then the cost of doing business is $\frac{1}{3}$ or $33\frac{1}{3}\%$ of the cost of goods, or $\frac{1}{5}$ or 20% of the sales.

304. Average Expenses. To be successful a merchant must keep a careful record of expenses and must reduce them to a minimum. The table below gives the average cost of doing business in different lines of retail trade and also the distribution of expenses, expressed as per cents of the total sales.*

	Dry Goods	Groceries	Hardware	Clothing	Furniture	Drugs	Shoes
Rent	3.1%	3.3%	2.9%	2.2%	5.5%	4.6%	3.1%
Salaries	9.6%	8.1%	11.2%	9.1%	8.3%	10.9%	11.1%
Advertising	1.5%	.7%	.9%	3.4%	2.9%	2.6%	1.5%
Heat and Light	.4%	.5%	.5%	.3%	.8%	.8%	.9%
Delivery	.9%	1.9%	.6%	.6%	1.1%	.4%	.3%
Supplies	.4%	.3%	.4%	.2%	.5%	.3%	.4%
Insurance and Taxes	1.1%	.4%	1.1%	1.1%	1.4%	1.4%	1.2%
General Expensees	4.4%	.3%	.8%	1.8%	1.8%	2.4%	4.1%
Depreciation and Shrinkage	1.4%	.6%	.5%	2.2%	1.9%	.5%	.6%
Bad Debts	.3%	.3%	.4%	.4%	1.2%	.2%	.1%
Total	23.1%	16.4%	19.3%	21.3%	25.4%	24.1%	23.3%

*From Sammon's Keeping Up With Rising Costs. Used by permission of The A. W. Shaw Company, publishers.

This table is compiled from data given by W. Sammons in *Keeping Up With Rising Costs,* based on an investigation of more than 1500 business houses.

PROBLEMS. FIGURING EXPENSES.

1. Find the total expenses of the following dry goods store. Find what per cent the expenses are of the total cost of the goods sold.

Rent	$ 1,780.00
Salaries	5,260.18
Advertising	739.84
Heat and Light	207.18
Delivery	465.81
Supplies	203.07
Insurance and Taxes	548.11
General Expenses	2,189.07
Depreciation	715.31
Bad Debts	417.49
Total cost of goods sold	$38,759.83
Total amount of sales	51,017.08

2. Find what per cent the total expenses of the above store are of the sales. How does this compare with the figure given in the table on page 353? Are the expenses of this store higher than they should be?

3. Find what per cent each item of expenses, of the above store, is of the total sales. Compare with the figures in the table on page 353. Along what lines should this store attempt to cut expenses?

IV. DETERMINING THE SELLING PRICE AND PROFIT.

305. General Discussion. Most of the older arithmetics give the impression that a merchant first buys, then sells and finally sits down and figures his profit or loss. As a matter of fact the figuring must be done before he sells the article or

even before he buys it. In some cases the retail merchant can not control either the cost or the selling price of an article as the selling price is fixed by the manufacturer. In such cases the only question for the merchant to decide is whether he can make a profit on the article. In deciding this he must take his expenses into consideration. If he can buy for $3.75 an article which he must sell for $5.00, he can make a gross profit of $1.25 or of 25% of the selling price. If his cost of doing business is 20% of his sales, he can make a net profit of 5% of the selling price, but if his cost of doing business is 25% of his sales, he can make no net profit.

In general, however, a merchant is free to fix the selling prices of the goods he handles, and he does this so as to insure him a reasonable net profit. Formerly it was the custom to express profits as a per cent of the cost price, but today many merchants express them as a per cent of the selling price. *To avoid confusion and mistakes the profit and cost of doing business should both be figured as per cents of the same thing.* If, as is sometimes done, expenses are figured as a per cent of the selling price and profits as a per cent of the cost the merchant, unless he is an expert mathematician, is very apt to fall into errors such as the following. Suppose a merchant whose expenses are 20% of his sales, sells for $10 an article which costs him $8. His gross profit is $2 or 25% of the cost of the article. The merchant is apt to think since his expenses are 20% and his gross profit 25%, that he is getting a net profit of 5% losing sight of the fact that he is figuring expenses and profits on a different basis. As a matter of fact he is getting no net profit as is seen if his profit is figured on the same base as his expenses. His gross profit is $2 which is only 20% of the selling price and as his expenses are also 20% of the selling price he is receiving no net profit.

306. Determining the Selling Price When Expenses and Profits are Expressed as Per Cents of Cost Price. The method

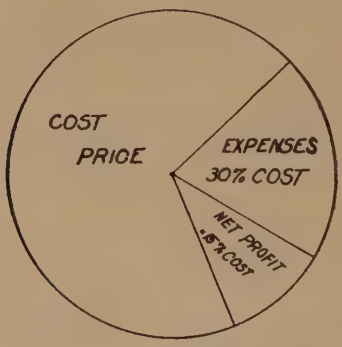
of finding the selling price that will give a certain per cent of profit when both expenses and profit are expressed as per cents of the cost is shown in the following example.

Illustrative Example. At what must a merchant sell an article costing him \$3.50 in order to receive a net profit of 15% of the cost, if his cost of doing business is 30% of the cost of the goods sold?

Solution.

Gross Profit=Expenses+Net Profit
=30% Cost+15% Cost=45% Cost
=.45×\$3.50=\$1.58

Selling Price=Cost+Gross Profit
=\$3.50+\$1.58=\$5.08



The relations between the Selling Price, Cost Price, Expenses, Net Profit and Gross Profit can be made clear by a diagram. The whole circle represents the Selling Price. It is evident from the diagram that the gross profit is 45% of the cost and that the selling price is 145% of the cost. The selling price may, therefore, be obtained by multiplying the cost price by 145% or 1.45.

307. Determining the Selling Price When Expenses and Profits are Expressed as Per Cents of Sales. The method of finding the selling price that will give a certain per cent of

profit when both expenses and profit are expressed as per cents of the sales is shown in the following example.

Illustrative Example. At what must a merchant sell an article costing him \$32.15 in order to receive a net profit of 10% of the selling price if his cost of doing business is 20% of his sales?

Solution.

Gross Profit=Expenses+Net Profit
=20% Selling Price+10% Selling Price
=30% Selling Price

Cost Price=Selling Price−Gross Profit
=Selling Price−30% Selling Price
=70% Selling Price

∴ 70% Selling Price=\$32.15
Selling Price=\$32.15÷.7=\$45.93

Again the relations between the Selling Price, Cost Price, Expenses, Gross Profit and Net Profit can be made clear by means of a diagram. The whole circle represents the Selling Price. From the diagram it is evident that the expenses and net profit amount to 30% of the selling price, so that 70% of the selling price must pay for the article or equal the cost price of \$32.15.

Problems. Selling Price and Profits.

1. A furniture dealer's expenses amount to 25% of the cost of the goods sold. He marks his goods to yield a net profit of 10% of the cost. At what must he mark a chair costing him $27.50?

2. Copy the following table and fill in the last three columns.

Cost of Article	Expenses	Net Profit Desired	Gross Profit Expressed as Per Cent of Cost	Selling Price Expressed as Per Cent of Cost	Selling Price in Dollars
$18.50	20% Cost	20% Cost	.....	.....	.....
32.25	25% Cost	15% Cost	.....	.....	.....
7.50	20% Cost	15% Cost	.....	.....	.....
65.00	30% Cost	15% Cost	.....	.....	.....
42.75	25% Cost	20% Cost	.....	.....	.....

3. The expenses of a certain shoe store are 23.2% of the sales. At what must they sell a pair of shoes costing them $5.50 in order to make a net profit of 10% of the selling price?

4. A certain make of safety razor sells at retail for $5. It costs the retailer $3.50. Can a druggist whose cost of doing business is 25% of his sales afford to sell this razor? His net profit will be what per cent of the selling price?

5. A certain make of electric washer is advertised to sell for $160. The manufacturers allow the retail merchant a discount of 33⅓% from the advertised price. What profit (expressed as a per cent of the selling price) can a hardware dealer make on this machine if his expenses are 20% of his sales? Solve this problem in the easiest way.

6. Copy the following table and fill in the other three columns.

Expenses	Net Profit Desired	Gross Profit Expressed as Per Cent of Sales	Cost Expressed as Per Cent of Selling Price	Cost of Article in Dollars	Selling Price
25% Sales	10% Sales	.....	.....	$35.18	.....
16% Sales	9% Sales	.....	.....	17.32	.....
27% Sales	10% Sales	.....	.....	82.15	.....
21% Sales	11% Sales	.....	.....	7.41	.....
23% Sales	10% Sales	.....	.....	27.57	.....

7. The expenses of a variety store in 1921 were $2,174.08. The total sales were $12,571.83. The expenses were what per cent of the sales?

8. At what price will this store have to sell the following articles in order to make a net profit of 9% of the selling price? (Use the result of the preceding problem).

Article	Cost
Rake	$ .80
Wash Basket	.60
Ironing Board	2.50
Step Ladder	2.15
Hammer	.40

9. The expenses of a certain furniture store in 1922 were $18,324.93. The total sales were $72,894.18. The expenses were what per cent of the sales?

10. At what price will this store have to sell the following articles in order to make a net profit of 8% of the selling price? (Use the result of the preceding problem.)

Article	Cost
Table	$12.50
Chair	8.71
Desk	38.50
Bed	21.80
Davenport	71.40

11. The store in the preceding problem sells on monthly payments and marks all goods at the payment price, allowing a discount of 20% from the marked price for cash. At what must they mark each of the articles in the preceding problem? See if you can find a short way of finding the marked price.

V. Collecting Money.

308. Statements. Both retailers and wholesalers usually send customers statements of their accounts on the first of each month. An illustration of such a statement is given below.

Goods are charged for the convenience of customers and itemized statements are mailed the first of every month. Please report any error immediately.

MONTHLY STATEMENT

A. Droney & Co.

HOWLING GREEN OHIO

Credit is extended with the understanding that Accounts are due within 60 days from date of purchase. Interest will be charged after that date. Please preserve this statement.

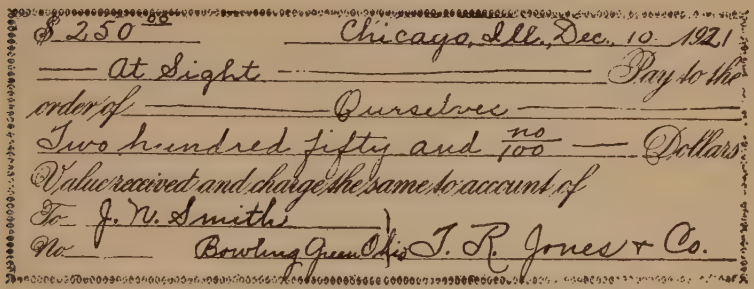
J R Averman

315 N Summit St - City

Statement of your Account for Month January 192x

DAY	DESCRIPTION		DEBIT	CREDIT	BALANCE
	Account Rendered				
10	10 yd. muslin	.20	2 00		
16	7 yd gingham	.25	1 75		
	½ yd gingham	.30	15		
19	1 doz. buttons		10		
	2 yd braid	15	30		
23	oxfords	8.00	8 00		
24	"			8 00	
27	1 veil		1 50		
			13 50	8 00	5 80

309. Commercial Drafts. If the retailer does not pay his account with the wholesaler when it is due the wholesaler may collect by drawing on the retailer through a bank. The *commercial draft* which is used for this purpose is a written demand on the debtor signed by the one to whom the money is due to pay the amount due.


250^{00} Chicago, Ill. Dec. 10 1921

At Sight Pay to the

order of Ourselves

Two hundred fifty and $^{no}/_{100}$ Dollars

Value received and charge the same to account of

To J. N. Smith

No. Bowling Green O. T. R. Jones & Co.

J. N. Smith of Bowling Green, Ohio, owes T. R. Jones and Co. of Chicago $250, so T. R. Jones and Co. make out the above draft and endorse it as follows:

Pay to the order of
The First National Bank of Chicago
for collection only
T. R. Jones and Co.

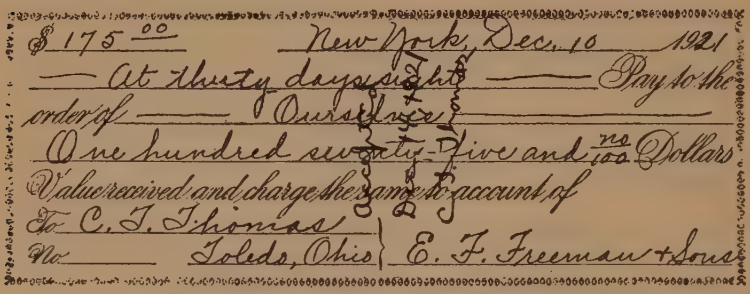
and give it to the First National Bank for collection. This bank endorses the draft as follows:

Pay to the order of
The Wood County Savings Bank of Bowling Green, Ohio
for collection only
The First National Bank of Chicago
E. E. French, Cashier.

and forwards it to the Bowling Green bank to collect. The Bowling Green bank presents it to Mr. Smith for collection. If Mr. Smith pays the draft, the money is forwarded to the

Chicago bank and paid to the credit of T. R. Jones and Co. This is an effective method of collecting money as the debtor does not want his bank to know that he owes a bill and refuses to pay it when due.

The draft illustrated above is called a *sight draft* and is used to collect money due. Wholesalers sometimes send a *time draft* with the invoice.


$175.00 New York, Dec. 10 1921
At thirty days sight Pay to the
order of Ourselves
One hundred seventy-five and no/100 Dollars
Value received and charge the same to account of
To C. T. Thomas
No Toledo, Ohio E. F. Freeman & Sons

If C. T. Thomas has bought a bill of foods amounting to $175 of E. F. Freeman & Sons, Terms: Thirty-day draft, he receives the above draft with the invoice, endorses it *accepted* as shown above and returns it to E. F. Freeman & Sons, who can borrow money from a bank giving the accepted draft as security. An accepted draft is called an *acceptance.*

Drafts are also used as a means of making C. O. D. shipments by freight. The shipper obtains an Order Bill of Lading from the railroad when shipping the goods and attaches it to a sight draft and sends both to a bank in the purchaser's town. In order to get the goods from the railroad the purchaser must have the bill of lading and to get the bill of lading he must pay the sight draft at the bank.

V. Bank Discount.

310. Borrowing Money on Notes and Accepted Drafts. As seen in the preceding section, a merchant holding a note or an accepted time draft can borrow money on it at a bank

giving the note or draft as security. When this is done the bank really buys the note or acceptance and will pay whatever it is worth when due, less the interest on the amount from the day the bank buys to the day it is due.

Retail merchants often accept notes from customers instead of cash in payment for goods and then sell the notes to their bank.

311. Definitions. Selling a note or acceptance to a bank is called *discounting* the note or acceptance. The interest charged by the bank is called *bank discount.* The amount the note is worth when due is called the *maturity value.* The amount received for the note, or the maturity value less the discount is called the *proceeds.* The bank charges interest on the note for the exact number of days from the day it is discounted until the day it matures. This time is called the *discount period.* The table on page 294 is used in determining this time.

312. Discounting Notes. The computing of bank discount is the same as the computing of interest and so involves nothing new. The following examples should make the method of procedure clear.

Example 1. If you hold a note for $300 dated Feb. 16, 1921, and payable in 90 days, what will you receive for it if you discount it on March 7, the rate of discount being 7%?

Solution.

Date of Maturity=May 17

Discount Period=March 7 to May 17=71 days

$$\text{Discount}=\frac{7}{\cancel{100}}\times\frac{71}{\cancel{360}\ 120}\times\frac{\cancel{300}\ 3}{}=\frac{497}{120}=\$4.14$$

Proceeds=Maturity Value−Discount=$300−$4.14=$295.86.

Example 2. If you hold a note for \$500 dated December 10, 1921, and payable in 3 months with 6% interest, what will you receive for it if you discount it on February 6, 1922, the rate of discount being 7%.

Solution.

$$\text{Interest}=\frac{\cancel{6}}{\cancel{100}}\times\frac{3}{\cancel{12}_{2}}\times\frac{\cancel{\$500}^{5}}{}=\frac{15}{2}=\$7.50$$

Maturity Value=\$500+\$7.50=\$507.50

Date of Maturity=March 10

Discount Period=Feb. 6 to March 10=32 days

$$\text{Discount}=\frac{7}{100}\times\frac{\cancel{32}^{4}}{\cancel{360}_{\cancel{45}_{9}}}\times\frac{\cancel{\$507.50}^{101.50}}{}=\frac{2842}{900}=\$3.16$$

Proceeds=Maturity Value—Discount

=\$507.50—\$3.16=\$504.34

Example 3. E. F. Freeman & Sons hold a thirty day sight draft on C. J. Thomas for \$175.00 accepted by Mr. Thomas on December 14, 1921. What will E. F. Freeman & Sons receive for this draft if they discount it on December 18 at 7%? (See draft on page 362).

Solution.

Date of Maturity=Jan. 13 (30 days after acceptance)

Discount Period=Dec. 18 to Jan. 13=26 days.

$$\text{Discount}=\frac{7}{100}\times\frac{\cancel{26}^{13}}{\cancel{360}_{\cancel{180}_{36}}}\times\frac{\cancel{\$175.00}^{35.00}}{}=\frac{3185}{3600}=\$0.88$$

Proceeds=Maturity Value—Discount

=\$175.00—\$0.88=\$174.12

Problems. Bank Discount.

1. On March 2, 1922, A. F. Smith gave his note for $250 to B. C. Jones, due in 90 days. Mr. Jones discounted the note at his bank on the same day, the rate of discount being 7%. What did Mr. Jones receive for the note? To whom will Mr. Smith pay the money due at the end of the 90 days? How much will Mr. Smith have to pay? What will the bank do if Mr. Smith fails to pay?

2. If Mr. Jones had kept the note in the preceding problem until April 15 what would he have received for it at the bank if the rate of discount was the same as before?

3. On June 3, M. O. Bates gave his note to D. R. Russell for $300 due in 3 months with interest at 7%. Mr. Russell sold the note to his bank on the same day, discount 7%. What did he receive?

4. If Mr. Russell had kept the note until August 11 what would he have received for it at the same rate of discount?

5. On May 1, Mr. G. A. Norton bought an automobile for $1250. In payment he gave the agent, A. Johnston and Co., $600 in cash and his note for the balance due in 90 days with 6% interest. On May 17, A. Johnston and Co. sold the note to their bank, the rate of discount being 6%. What did they get for the note?

6. If E. F. Freeman and Sons discount the accepted draft shown on page 362 at their bank on December 17, what will they receive for it, the rate of discount being 7%.

7. Find the net proceeds on each of the following.

Face of Note	Date of Note	Time of Note	Rate of Interest	Date of Discount	Rate of Discount
$850	Jan. 7	60 days	none	Jan. 7	6%
400	Aug. 13	3 months	6%	Sept. 1	7%
900	May 5	90 days	none	May 5	7%
650	July 29	4 months	7%	Sept. 1	7%

APPENDIX

Tables for Reference

Length

12 inches (in.)=1 foot (ft.)
3 feet=1 yard (yd.)
5½ yards, or 16½ feet=1 rod (rd.)
320 rods, or 5280 feet=1 mile (mi.)
3 ft. 8 in. is often written 3′ 8″.

Square Measure

144 square inches (sq. in.)=1 square foot (sq. ft.)
9 square feet=1 square yard (sq. yd.)
30¼ square yards=1 square rod (sq. rd.)
160 square rods=1 acre (A.)
640 acres=1 square mile (sq. mi.)

Cubic Measure

1728 cubic inches (cu. in.)=1 cubic foot (cu. ft.)
27 cubic feet=1 cubic yard (cu. yd.)

Avoirdupois Weight

16 ounces (oz.)=1 pound (lb.)
2000 pounds=1 ton (T.)

Liquid Measure

4 gills (gi.)=1 pint (pt.)
2 pints=1 quart (qt.)
4 quarts=1 gallon (gal.)
31½ gallons=1 barrel (bbl.)
63 gallons=1 hogshead (hhd.)
1 gallon=231 cu. in.
1 cu. ft.=7½ gal. (approx.)

Dry Measure

2 pints (pt.)=1 quart (qt.)
8 quarts=1 peck (pk.)
4 pecks=1 bushel (bu.)
A bushel=2150.42 cu. in., or 1¼ cu. ft. (approx.)

Time Measure

60 seconds (sec.)=1 minute (min.)
24 hours=1 day (da.)
60 minutes=1 hour (hr.)
7 days=1 week (wk.)
About 30 days=1 month (mo.)
365 days=1 common year (yr.)
366 days=1 leap year

Angle and Arc Measure

60 seconds (″)=1 minute (′)
60 minutes=1 degree (°)
360 degrees=4 right angles

Numbers	Paper
12 units=1 dozen (doz.)	24 sheets=1 quire
12 dozen=1 gross (gr.)	20 quires=1 ream
12 gross=1 great gross	2 reams=1 bundle
20 units=1 score	5 bundles=1 bale

For convenience in counting, 500 sheets are commonly called a ream.

METRIC TABLES

Metric Length

A kilometer (km.)=1,000 meters
A hektometer=100 meters
A dekameter=10 meters
Meter (m.)
A decimeter (dm.)=0.1 of a meter
A centimeter (cm.)=0.01 of a meter
A millimeter (mm.)=0.001 of a meter

Metric Square Measure

A square kilometer (sq. km.)=1,000,000 square meters
A square hektometer=10,000 square meters
A square dekameter=100 square meters
Square meter (sq. m.)
A square decimeter=0.01 of a square meter
A square centimeter (sq. cm.)=0.0001 of a square meter
A square millimeter (sq. mm.)=0.000001 of a square meter

Metric Cubic Measure

A cubic hektometer=1,000,000 cubic meters
A cubic dekameter=1,000 cubic meters
Cubic meter (cu. m.)
A cubic decimeter (cu. dm.)=0.001 of a cubic meter
A cubic centimeter (cu. cm.)=0.000001 of a cubic meter
A cubic millimeter (cu. mm.)=0.000000001 of a cubic meter

Metric Capacity.

A hektoliter (hl.)=100 liters
A dekaliter=10 liters
Liter (l.)
A deciliter (dl.)=0.1 of a liter
A centiliter (cl.)=0.01 of a liter
A milliliter (ml.)=0.001 of a liter

Metric Weight

A kilogram (kg.)=1000 grams
Gram (g.)
A decigram (dg.)=0.1 of a gram
A centigram (cg.)=0.01 of gram
A milligram (mg.)=0.001 of a gram

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for 60 days, more pent to lift two pie

$70.12, 60 day pencils = $14.10

30 days = 1/2 a 60 day pencil. $3.5-

6 day pencil = $1.

15 days = 2 2/3 times a 6 day pencil =

State Board of Ed

N. C. Cook – Killen

Bernard McClarty – Bluefield

Lena Lowe Yost – Huntington

Mr McLaughlin – Charleston

Mr. Gorby – New Martinsville

Geo. M. Ford – President

Frank Marsh – Secretary

Area of a circle = $\frac{1}{2}$ C r ($\frac{1}{2}$ C x r)

Circumference = π d (3.1416 x d).

Diameter = 2 r (2 x r)

Circumference = C = 2 π r (2 x 3.1416 x 2

Area of a circle = πr^2 (3.1416 x r^2)

Square Root. Hcf

u u u u

12 10 + 2

12 10 + 2

2 4 20 + 4

2

100 + 20

144

100 + 40 + 4 units = 144 units

Exponent - the small figure 2 or 3 a little above a number to show that it is to be squared or cubed.

square root of a number is the reverse to the square of a number.

tens = 10

unit = 2

t + u

t + u

tu + u^2

tu